

The Weak Field Limit of Fourth Order Gravity

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Fourth Order Theories of Gravity have recently attracted a lot of interest as candidates to explain the observed cosmic acceleration, the flatness of the rotation curves of spiral galaxies, the large scale structure and other relevant astrophysical phenomena. This means that the "Dark Side" issue of the Universe could be completely reversed considering dark matter and dark energy as "shortcomings" of General Relativity in its simplest formulation (a linear theory in the Ricci scalar R , minimally coupled to the standard perfect fluid matter) and claiming for the "correct" theory of Gravity as that derived by matching the largest number of observational data, without imposing any theory a priori. As a working hypothesis, accelerating behavior of cosmic fluid, large scale structure, potential of galaxy clusters, rotation curves of spiral galaxies could be reproduced by means of extending General Relativity to generic actions containing higher order and non-minimally coupled terms in curvature invariants. In other words, gravity could act in different ways at different scales and the above "shortcomings" could be due to the incorrect extrapolations of the Einstein theory, actually tested at short scales and low energy regimes. Very likely, what we call "dark matter and "dark energy" could be nothing else but signals of the breakdown of General Relativity at large scales. Then, it is a crucial point testing these Extended Theories in the weak field limit. In this sense, comparing these theories to General Relativity could be a fundamental step to retain or rule out them. In this review paper, after a survey of what is intended for Extended Theories of Gravity in the so called *metric approach*, we extensively discuss their Newtonian and the post-Newtonian limits pointing out, in details, their resemblances and differences with respect to General Relativity. Particular emphasis is placed on the exact solutions and methods used to obtain them. Finally, it is clearly shown that General Relativity results, in the Solar System context, are easily recovered since Einstein theory is a particular case of this extended approach. This is a crucial point against several wrong results in literature stating that these theories (e.g. $f(R)$ -gravity) are not viable at local scales.

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I. INTRODUCTION

In recent years, the effort to give a physical explanation to the today observed cosmic acceleration [1] has attracted a good amount of interest in Fourth Order Gravity (FOG) considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations without the introduction of dark matter and dark energy. At fundamental level, several efforts have been aimed towards the unification of gravity with the other interactions of physics, like Electromagnetism, assuming GR as the only fundamental theory capable of explaining the gravitational interaction. The failure of such attempts led to the common belief that GR had to be revised in the ultraviolet limit in order to address issues like quantization and renormalization. These are only some aspects of the several physical and mathematical motivations to enlarge GR to more general approaches. For comprehensive reviews of the problem, see [2–7].

Other issues come from astrophysics. For example, the observed Pioneer anomaly problem [8] can be framed into the same approach [9] and then, apart the cosmology and quantum field theory, a systematic analysis of such theories urges at small, medium and large scales. In particular, a delicate point is to address the weak field limit of any extended theory of gravity since two main issues are extremely relevant: *i*) preserving the results of GR at local scales since they well fit Solar System experiments and observations; *ii*) enclosing in a self-consistent and comprehensive picture phenomena as anomalous acceleration or dark matter at Galactic scales.

It is straightforward to extend GR to theories with additional geometric degrees of freedom and several recent proposals focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading

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to fourth-order and higher-order field equations [10–13]. Such an approach has become a sort of paradigm in the study of gravitational interaction consisting, essentially, *in adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of some unification or quantum gravity theory.*

The idea to extend Einstein's theory of gravitation is fruitful and economic also with respect to several attempts which try to solve problems by adding new and, most of times, unjustified ingredients in order to give self-consistent pictures of dynamics. The today observed accelerated expansion of the Hubble flow and the missing matter at astrophysical scales are primarily enclosed in these considerations. Both the issues could be solved by changing the gravitational sector, *i.e.* the *l.h.s.* of field equations. The philosophy is alternative to add new cosmic fluids (new components in the *r.h.s.* of field equations) which should give rise to clustered structures (dark matter) or to accelerated dynamics (dark energy) thanks to exotic equations of state. In particular, relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar R , like in the Hilbert-Einstein formulation, one can take into account an effective action where the gravitational Lagrangian includes other scalar invariants.

Due to the increased complexity of the field equations, the main body of theoretical works dealt with the effort to achieve some formally equivalent theories which could be handled in a simpler way. In this sense, a reduction of the differential order of the field equations can be achieved by considering metric and connection as independent objects in the so called *Palatini approach* [14, 15] and energy conditions have to be carefully discussed [16].

In addition, other authors exploited the formal relations to Scalar-Tensor theories to make some statements about the weak field regime [18], which was already worked out for scalar-tensor theories [19]. Also a post-Newtonian parameterization with metric approach in the Jordan Frame has been considered [20].

In this review paper, we want to address the general problem of the weak field limit for theories of gravity where higher order curvature invariants are present. In particular, we deal with theories where Riemann tensor, Ricci tensor, and Ricci scalar are considered in the effective action. We deduce the field equations, discuss the weak field limit and derive the weak field potentials with corrections to the Newtonian potential. The plan of the paper is the following:

In *section II*, we provide a short summary of GR and discuss its extension to FOG. Besides we take into account the conformal transformations showing how such theories can be discussed under the standard of scalar-tensor gravity.

In *section III*, we give generalities on spherically symmetric, Birkhoff theorem, Eddington parameters and deviations from GR solutions. A general perturbation approach for $f(R)$ -gravity is discussed starting from GR.

Section IV is devoted to the technical development of field equations with respect to Newtonian and post-Newtonian approaches.

In *section V* we take into account the debate on the analogy between $f(R)$ - gravity and Scalar-Tensor gravity. We show that $f(R)$ -models are dynamically equivalent to O'Hanlon models which is a special case of Scalar-Tensor gravity characterized by a self-interaction potential without a kinetic term. We show that comparing the results of this theory with GR, in the weak field limit, could lead to misleading conclusions.

In *section VI*, we analyze the Newtonian limit of $f(R)$ -Gravity. We are going to focus exclusively on the weak field limit within the metric approach. By using the development for a generic analytic function $f(R)$, we find the solution in the vacuum with standard coordinates. Besides, we show that the Birkhoff theorem is not a general result for $f(R)$ -gravity since time-dependent evolution for spherically symmetric solutions can be achieved according to the order of perturbation. In other words, solutions could not be stable as requested by the Birkhoff theorem.

In *section VII*, we report the Newtonian and post-Newtonian limit of field equations by using the Green function method for $f(R)$ -gravity and the formal solutions in the harmonic gauge. A general discussion about the mathematical properties of equations, the solutions and their deviations with respect to Gauss and Birkhoff theorem, and the Minkowskian behavior of metric tensor are reported. We derive the general solutions (at Newtonian and post-Newtonian levels) when an uniform massive spherical source is considered. The point-like source limit of the Newtonian solution is discussed and the compatibility of $f(R)$ -gravity with respect to GR is shown.

In *section VIII*, we discuss the Newtonian limit but for a *quadratic gravity Lagrangian* where quadratic curvature invariants are presents. Also in this case, we adopt the Green function method. We find the internal and external potential generated by an extended spherically symmetric matter source. A particularly detailed discussion about the field equations and their solutions is provided for a set of values of the arbitrary constants present in the model.

In *section IX* the Newtonian limit of the most general FOG is discussed with no gauge condition. The most general theory with fourth order differential equations is obtained by generalizing the $f(R)$ -models in the action with a generic function containing the other two curvature invariants: *Ricci square* ($R_{\alpha\beta}R^{\alpha\beta}$) and *Riemann square* ($R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$). Considering the Gauss - Bonnet invariant, it is possible to show that only two of these invariants are really necessary.

In *section X*, we draw the conclusions and discuss the possible applications of the results.

II. GENERAL RELATIVITY AND ITS EXTENSIONS

Any relativistic theory of gravity has to match some minimal requirements to address gravitational dynamics. First of all, it has to explain issues coming from Celestial Mechanics as the planetary orbits, the potential of self-gravitating systems, the Solar System stability.

This means that it has to reproduce the Newtonian dynamics in the weak field limit and then it has to pass the Solar System experiments which are all well founded and constitute the test bed of GR [21].

Besides, any theory of gravity has to be consistent with stellar structures and galactic dynamics considering the observed baryonic constituents (e.g. luminous components as stars, sub-luminous components as planets, dust and gas), radiation and Newtonian potential which is, by assumption, extrapolated to galactic scales.

The third step is cosmology and large scale structure which means to reproduce, in a self-consistent way, the cosmological parameters as the expansion rate, the Hubble constant, the density parameter and the clustering of galaxies. Observations probe the standard baryonic matter, the radiation and an attractive overall interaction, acting at all scales and depending on distance. From a phenomenological point of view this is *gravity*.

GR is the simplest theory which partially satisfies the above requirements [22]. It is based on the assumption that space and time are entangled into a single spacetime structure, which, in the limit of no gravitational forces, has to reproduce the Minkowski spacetime. Besides, the Universe is assumed to be a curved manifold and the curvature depends on mass-energy distribution [23]. In other words, the distribution of matter influences point by point the local curvature of the spacetime structure.

Furthermore, GR is based on three first principles that are *Relativity*, *Equivalence*, and *General Covariance* (see [4, 7, 24] for detailed discussions). Another requirement is the *Principle of Causality* that means that each point of spacetime admits a notion of past, present and future.

Let us also recall that the Newtonian theory, the weak field limit of GR, requires absolute concepts of space and time, that particles move in a preferred inertial frame following curved trajectories function of the sources (*i.e.*, the "forces").

On these bases, GR postulates that gravitational forces have to be expressed by the curvature of a metric tensor field $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$ on a four-dimensional spacetime manifold, having the same signature of Minkowski metric, here assumed to be $(+ - - -)$. Curvature is locally determined by the distribution of the sources, that is, being the spacetime a continuum, it is possible to define a stress-energy tensor $T_{\mu\nu}$ which is the source of the curvature.

Once a metric $g_{\mu\nu}$ is given, the inverse $g^{\mu\nu}$ satisfies the condition¹

$$g^{\mu\alpha}g_{\alpha\beta} = \delta^\mu_\beta \quad (1)$$

Its curvature is expressed by the *Riemann tensor* (curvature)

$$R^\alpha_{\mu\beta\nu} = \Gamma^\alpha_{\mu\nu,\beta} - \Gamma^\alpha_{\mu\beta,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\alpha_{\sigma\beta} - \Gamma^\sigma_{\mu\beta}\Gamma^\alpha_{\sigma\nu} \quad (2)$$

where the comas are partial derivatives. The $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}) \quad (3)$$

if the Levi-Civita connection is assumed. The contraction of the Riemann tensor (2)

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = \Gamma^\sigma_{\mu\nu,\sigma} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho} - \Gamma^\rho_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \quad (4)$$

is the *Ricci tensor* and the scalar

$$R = g^{\sigma\tau}R_{\sigma\tau} = R^\sigma_{\sigma} = g^{\tau\xi}\Gamma^\sigma_{\tau\xi,\sigma} - g^{\tau\xi}\Gamma^\sigma_{\tau\sigma,\xi} + g^{\tau\xi}\Gamma^\sigma_{\tau\xi}\Gamma^\rho_{\sigma\rho} - g^{\tau\xi}\Gamma^\rho_{\tau\sigma}\Gamma^\sigma_{\xi\rho} \quad (5)$$

is called the *scalar curvature* of $g_{\mu\nu}$. The Riemann tensor (2) satisfies the so-called *Bianchi identities* and the *contracted Bianchi identities*, that is

¹ The Greek index runs between 0 and 3; the Latin index between 1 and 3.

$$\left\{ \begin{array}{l} R_{\alpha\mu\beta\nu;\delta} + R_{\alpha\mu\delta\beta;\nu} + R_{\alpha\mu\nu\delta;\beta} = 0 \\ R_{\alpha\mu\beta\nu}{}^{;\alpha} + R_{\mu\beta;\nu} - R_{\mu\nu;\beta} = 0 \\ 2R_{\alpha\beta}{}^{;\alpha} - R_{;\beta} = 0 \\ 2R_{\alpha\beta}{}^{;\alpha\beta} - \square R = 0 \end{array} \right. \quad (6)$$

where the covariant derivative is $A^{\alpha\beta\dots\delta}{}_{;\mu} = \nabla_\mu A^{\alpha\beta\dots\delta} = A^{\alpha\beta\dots\delta}{}_{,\mu} + \Gamma_{\sigma\mu}^\alpha A^{\sigma\beta\dots\delta} + \Gamma_{\sigma\mu}^\beta A^{\alpha\sigma\dots\delta} + \dots + \Gamma_{\sigma\mu}^\delta A^{\alpha\beta\dots\sigma}$ and $\nabla_\alpha \nabla^\alpha = \square = \frac{\partial_\alpha(\sqrt{-g}g^{\alpha\beta}\partial_\beta)}{\sqrt{-g}}$ is the d'Alembert operator with respect to the metric $g_{\mu\nu}$ (see for the details [25]).

Assuming that matter is given as a perfect fluid, that is

$$T_{\mu\nu} = (p + \rho)u_\mu u_\nu - p g_{\mu\nu} \quad (7)$$

where $u_\mu u_\nu$ defines a comoving observer (with the conditions $g^{tt}u_t^2 = 1$, $u_i = 0$ where $x^0 = ct$ and we are assuming natural units with $c = 1$; p is the pressure and ρ the mass-energy density of the fluid, then the continuity equation requires $T_{\mu\nu}$ to be covariantly constant, *i.e.* it has to satisfy the conservation law

$$T^{\mu\sigma}{}_{;\sigma} = 0 \quad (8)$$

that are nothing else but contracted Bianchi identities. The GR field equations are then

$$G_{\mu\nu} = \mathcal{X} T_{\mu\nu} \quad (9)$$

where

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} \quad (10)$$

is the "Einstein tensor" of $g_{\mu\nu}$. These equations are both variational and satisfy the conservation laws (8) since the following relation holds

$$G^{\mu\sigma}{}_{;\sigma} = 0 \quad (11)$$

as a byproduct of the above Bianchi identities [25, 26].

The Hilbert-Einstein Lagrangian that allows to obtain the field equations (9) is the sum of an ordinary *matter Lagrangian* \mathcal{L}_m (minimally coupled) and of the Ricci scalar:

$$\mathcal{L}_{HE} = \sqrt{-g}(R + \mathcal{X}\mathcal{L}_m) \quad (12)$$

where $\sqrt{-g}$ denotes the square root of the value of the determinant of the metric $g_{\mu\nu}$ and the coupling constant is $\mathcal{X} = 8\pi G$. The action of GR is

$$\mathcal{A}^{GR} = \int d^4x \sqrt{-g}(R + \mathcal{X}\mathcal{L}_m) \quad (13)$$

From the action principle, we get the field equations (9) by the variation:

$$\delta\mathcal{A}^{GR} = \delta \int d^4x \sqrt{-g}(R + \mathcal{X}\mathcal{L}_m) = \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{R}{2}g_{\mu\nu} - \mathcal{X}T_{\mu\nu} \right] \delta g^{\mu\nu} + \int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = 0 \quad (14)$$

where $T_{\mu\nu}$ is energy momentum tensor of matter

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_m)}{\delta g^{\mu\nu}} \quad (15)$$

The last term in (14) is a 4-divergence

$$\int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} [(-\delta g^{\mu\nu})_{;\mu\nu} - \square(g^{\mu\nu} \delta g_{\mu\nu})] \quad (16)$$

then we can neglect it and we get the field equation (9). For the variational calculus (14), the following relations can be used

$$\begin{cases} \delta(g_{\mu\nu} g^{\mu\nu}) = g_{\mu\nu} \delta g^{\mu\nu} + g^{\mu\nu} \delta g_{\mu\nu} = 0 \\ \delta\sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta} \\ \delta R = R_{\alpha\beta} \delta g^{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta} \\ \delta R_{\alpha\beta} = \delta g^\rho_{(\alpha;\beta)\rho} - \frac{1}{2} \square \delta g_{\alpha\beta} - \frac{1}{2} g^{\rho\sigma} \delta g_{\rho\sigma;\alpha\beta} \end{cases} \quad (17)$$

The choice by Hilbert and Einstein was completely arbitrary but it was the simplest one both from the mathematical and the physical viewpoint. As it was later clarified by Levi-Civita curvature is not a "purely metric notion" but, rather, a notion related to the "linear connection" to which "parallel transport" and "covariant derivation" refer [27].

This is the precursor idea of the "gauge theoretical framework" [28], defined after the pioneering work by Cartan [29].

It was later clarified that the principles of relativity, equivalence, covariance, and causality, just require that the spacetime structure has to be determined by either one or both of two fields, a Lorentzian metric g and a linear connection Γ , assumed to be torsionless for the sake of simplicity.

The metric g fixes the causal structure of spacetime (the light cones) as well as its metric relations (clocks and rods); the connection Γ fixes the free-fall, *i.e.* the locally inertial observers. They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow the null geodesics of Γ , so that Γ and g can be independent, *a priori*, but constrained, *a posteriori*, by some physical restrictions. These, however, do not impose that Γ has necessarily to be the Levi-Civita connection of g [30].

This achievement justifies the fact, at least in principle, "*Extended Theories of Gravity*" can be considered starting from the same points by Einstein and Hilbert: they are theories in which gravitation is described by either a metric (the so-called "purely metric theories"), or by a linear connection (the so-called "purely affine theories") or by both fields (the so-called "metric-affine theories", also known as "first order formalism theories"). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both g and Γ .

The choice (12) is by no means unique and it turns out that the Hilbert-Einstein Lagrangian is in fact the only choice that produces an invariant that is linear in second derivatives of the metric (or first derivatives of the connection). A Lagrangian that, unfortunately, is rather singular from the Hamiltonian viewpoint, in much the same way as Lagrangians, linear in canonical momenta, are rather singular in Classical Mechanics (see *e.g.* [31]).

A number of attempts to generalize GR (and unify it to Electromagnetism) along these lines were followed by Einstein himself and many others (Eddington, Weyl, Schrodinger, just to quote the main contributors; see, *e.g.*, [32]) but they were given up because of a number of difficulties related to the complicated structure of a non-linear theory (where by "non-linear" we mean here a theory that is based on non-linear invariants of the curvature tensor), and also because of the new understanding of physics that is currently based on four fundamental forces and requires a general "gauge framework" to be adopted (see [33]).

Further curvature invariants or non-linear functions of them should be also considered, especially in view of the fact that they have to be included in both the semi-classical expansion of any quantum Lagrangian or in the low-energy limit of a string Lagrangian.

Moreover, it is clear from the recent astrophysical observations that Einstein equations are no longer a good test for gravitation at all scales, that is at Solar System, galactic, extra-galactic and cosmic scales, unless one does not admit that the *r.h.s.* of Eqs.(9) contains some kind of exotic matter-energy density which are generically addressed as the "dark matter" and "dark energy".

From our point of view, the philosophy of Extended Theories of Gravity is much simpler [7]. Instead of changing the matter side of Einstein Equations (9) in order to fit the "missing matter-energy" content of the currently observed

Universe by adding new forms of matter and energy, we assume that it is more convenient to change the gravitational side of the equations, admitting corrections coming from non-linear Lagrangians. However such an approach should be explored. Of course, the Lagrangians should be conveniently tuned up on the basis of their best fit with all possible observational tests, at infrared scales (Solar, galactic, extragalactic and cosmic) and on the basis of consistency with fundamental theories at ultraviolet scales.

Let us take into account an important class of Extended Theories of Gravity, the Fourth Order Gravity. It is the first straightforward generalization of GR and can be directly related to the ultraviolet (quantum gravity) and infrared (cosmology) issues of any final comprehensive theory of gravity.

A. Fourth Order Gravity

Let us start with the general class of *Fourth Order Gravity* (FOG) given by the action

$$\mathcal{A}^f = \int d^4x \sqrt{-g} \left[f(X, Y, Z) + \mathcal{X} \mathcal{L}_m \right] \quad (18)$$

where f is an unspecified analytical function of curvature invariants $X = R$ (Ricci scalar), $Y = R_{\alpha\beta} R^{\alpha\beta}$ (Ricci tensor square) and $Z = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ (Riemann square). By varying the action (18) in the metric approach and by using the properties (17) with following additional properties

$$\begin{cases} \delta Y = \delta(R_{\alpha\beta} R^{\alpha\beta}) = 2R_{\mu}^{\alpha} R_{\alpha\nu} \delta g^{\mu\nu} + 2R^{\mu\nu} \delta R_{\mu\nu} \\ \delta Z = \delta(R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}) = 4R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} \delta g^{\mu\nu} + 2R^{\alpha\beta\gamma\delta} \delta R_{\alpha\beta\gamma\delta} \\ \delta R_{\alpha\beta\gamma\delta} = \delta(g_{\alpha\sigma} R^{\sigma}_{\beta\gamma\delta}) = R^{\sigma}_{\beta\gamma\delta} \delta g_{\alpha\sigma} + g_{\alpha\sigma} \delta R^{\sigma}_{\beta\gamma\delta} \\ \delta R^{\alpha}_{\beta\gamma\delta} = \delta g^{\alpha}_{(\beta;\delta)\gamma} - \delta g^{\alpha}_{(\beta;\gamma)\delta} + \delta g_{\beta[\gamma}^{\alpha}_{\delta]} \end{cases} \quad (19)$$

we have

$$\begin{aligned} \delta \mathcal{A}^f &= \delta \int d^4x \sqrt{-g} [f(X, Y, Z) + \mathcal{X} \mathcal{L}_m] \\ &= \int d^4x \sqrt{-g} \left[\left(f_X R_{\mu\nu} + 2f_Y R_{\mu}^{\alpha} R_{\alpha\nu} + 4f_Z R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} \right) \delta g^{\mu\nu} \right. \\ &\quad \left. + \left(g^{\mu\nu} f_X + 2f_Y R^{\mu\nu} \right) \delta R_{\mu\nu} + 2f_Z R^{\alpha\beta\gamma\delta} \delta R_{\alpha\beta\gamma\delta} \right] \\ &= \int d^4x \sqrt{-g} \left\{ \left(f_X R_{\mu\nu} + 2f_Y R_{\mu}^{\alpha} R_{\alpha\nu} + 4f_Z R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} \right) \delta g^{\mu\nu} \right. \\ &\quad + f_X [-(\delta g^{\mu\nu})_{;\mu\nu} - \square(g^{\mu\nu} \delta g_{\mu\nu})] + f_Y R^{\mu\nu} [2\delta g^{\rho}_{(\mu;\nu)\rho} - \square \delta g_{\mu\nu} - g^{\rho\sigma} \delta g_{\rho\sigma;\mu\nu}] \\ &\quad \left. + 2f_Z R^{\alpha\beta\gamma\delta} [\delta g_{\alpha(\beta;\delta)\gamma} - \delta g_{\alpha(\beta;\gamma)\delta} + \delta g_{\beta[\gamma;\alpha\delta]}] \right\} \\ &\sim \int d^4x \sqrt{-g} \left\{ f_X R_{\mu\nu} + 2f_Y R_{\mu}^{\alpha} R_{\alpha\nu} + 2f_Z R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu} \square f_X \right. \\ &\quad \left. - 2[f_Y R^{\alpha}_{(\mu);\nu)\alpha} + \square[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{\alpha\beta} g_{\mu\nu} - 4[f_Z R_{\mu}^{\alpha\beta}_{\nu}]_{;\alpha\beta} \right\} \delta g^{\mu\nu} \\ &= \int d^4x \sqrt{-g} (H_{\mu\nu} - \mathcal{X} T_{\mu\nu}) \delta g^{\mu\nu} = 0 \end{aligned} \quad (20)$$

where $f_X = \frac{df}{dX}$, $f_Y = \frac{df}{dY}$, $f_Z = \frac{df}{dZ}$ and the symbol \sim means that we neglected pure divergences. Then the field equations of FOG are

$$\begin{aligned}
H_{\mu\nu} = & f_X R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - f_{X;\mu\nu} + g_{\mu\nu} \square f_X + 2f_Y R_{\mu}^{\alpha} R_{\alpha\nu} - 2[f_Y R^{\alpha}_{(\mu};\nu)_{\alpha} + \square[f_Y R_{\mu\nu}] + [f_Y R_{\alpha\beta}]^{\alpha\beta} g_{\mu\nu} \\
& + 2f_Z R_{\mu\alpha\beta\gamma} R_{\nu}^{\alpha\beta\gamma} - 4[f_Z R_{\mu}^{\alpha\beta}{}_{\nu}]_{;\alpha\beta} = \mathcal{X} T_{\mu\nu}
\end{aligned} \tag{21}$$

The trace of field equations (21) is the following

$$H = g^{\alpha\beta} H_{\alpha\beta} = f_X R + 2f_Y R_{\alpha\beta} R^{\alpha\beta} + 2f_Z R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2f + \square[3f_X + f_Y R] + 2[(f_Y + 2f_Z) R^{\alpha\beta}]_{;\alpha\beta} = \mathcal{X} T \tag{22}$$

where $T = T^{\sigma}_{\sigma}$ is the trace of energy-momentum tensor. Moreover the (21) satisfies the condition $H^{\alpha\mu}_{;\alpha} = \mathcal{X} T^{\alpha\mu}_{;\alpha} = 0$. In fact it is easy to check (in the case of $f(R)$ -Gravity) that

$$\begin{aligned}
H^{\alpha\mu}_{;\alpha} = & f'_{;\alpha} R^{\alpha\mu} + f' R^{\alpha\mu}_{;\alpha} - \frac{1}{2} f^{;\mu} - f'^{\alpha\mu}_{;\alpha} + f'^{\alpha}_{\alpha}{}^{\mu} = f'' R^{\alpha\mu} R_{;\alpha} - f'^{\alpha\mu}_{;\alpha} + f'^{\alpha}_{\alpha}{}^{\mu} = \\
& f'' R^{\alpha\mu} R_{;\alpha} - f'^{\alpha}_{\alpha}{}^{\mu} = f'' R^{\alpha\mu} R_{;\alpha} - f'' R^{\alpha}_{\alpha}{}^{\mu} = 0
\end{aligned} \tag{23}$$

where we used the properties $G^{\alpha\mu}_{;\alpha} = 0$ and $[\nabla^{\mu}, \nabla_{\alpha}] f'^{\alpha} = -f'^{\alpha}_{\alpha}{}^{\mu}$.

B. Conformal transformations

The above theories can be easily viewed as scalar-tensor theories of gravity (for a comprehensive discussion see [7]). Let us now introduce conformal transformations to show that any scalar-tensor theory, in absence of ordinary matter, e.g. a perfect fluid, is conformally equivalent to an Einstein theory plus minimally coupled scalar fields. If standard matter is present, conformal transformations allow to transfer non-minimal coupling to the matter component [34, 35]. A general non-minimally coupled scalar-tensor theory is

$$\mathcal{A}^{ST} = \int d^4x \sqrt{-g} [F(\phi)R + \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} + V(\phi) + \mathcal{X}\mathcal{L}_m] \tag{24}$$

where $V(\phi)$ and $F(\phi)$ are generic functions describing the self-interaction potential and the coupling of a scalar field ϕ , respectively. The Brans-Dicke theory of gravity is a particular case of the action (24) in which $V(\phi) = 0$ and $\omega(\phi) = -\frac{\omega_{BD}}{\phi}$. In fact we have

$$\mathcal{A}^{BD} = \int d^4x \sqrt{-g} \left[\phi R - \omega_{BD} \frac{\phi_{;\alpha}\phi^{;\alpha}}{\phi} + \mathcal{X}\mathcal{L}_m \right] \tag{25}$$

The variation of (24) with respect to $g_{\mu\nu}$ and ϕ gives the second-order field equations

$$\begin{cases}
F(\phi)G_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu} + \omega(\phi)\left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}\phi_{;\alpha}\phi^{;\alpha}g_{\mu\nu}\right] - F(\phi)_{;\mu\nu} + g_{\mu\nu}\square F(\phi) = \mathcal{X}T_{\mu\nu} \\
2\omega(\phi)\square\phi - \omega_{,\phi}(\phi)\phi_{;\alpha}\phi^{;\alpha} - [F(\phi)R + V(\phi)]_{;\phi} = 0 \\
3\square F(\phi) - F(\phi)R - 2V(\phi) - \omega(\phi)\phi_{;\alpha}\phi^{;\alpha} = \mathcal{X}T \\
2\omega(\phi)\square\phi + 3\square F(\phi) - [\omega_{,\phi}(\phi) + \omega(\phi)]\phi_{;\alpha}\phi^{;\alpha} - [F(\phi)R + V(\phi)]_{;\phi} - F(\phi)R - 2V(\phi) = \mathcal{X}T
\end{cases} \tag{26}$$

The third equation in (26) is the trace of the field equation for $g_{\mu\nu}$ and the last is a combination of the trace and of the field equation for ϕ .

The conformal transformation on the metric $g_{\mu\nu}$ is

$$\tilde{g}_{\mu\nu} = A(x^{\lambda})g_{\mu\nu} \tag{27}$$

with $A(x^\lambda) > 0$. A is the conformal factor. Obviously the transformation rule for the contravariant metric tensor is $\tilde{g}^{\mu\nu} = A^{-1}g^{\mu\nu}$. The mathematical quantities in the Einstein frame (EF) (quantities referred to $\tilde{g}_{\mu\nu}$) are linked to the ones in the Jordan Frame (JF) (quantities referred to $g_{\mu\nu}$ and action (24)) as follows

$$\left\{ \begin{array}{l} \tilde{\Gamma}_{\mu\nu}^\alpha = \Gamma_{\mu\nu}^\alpha + \phi_{;\mu}\delta_\nu^\alpha + \phi_{;\nu}\delta_\mu^\alpha - \phi^{;\alpha}g_{\mu\nu} \\ \tilde{R}^\alpha_{\mu\beta\nu} = R^\alpha_{\mu\beta\nu} - \delta_\beta^\alpha(\phi_{;\mu\nu} - \phi_{;\mu}\phi_{;\nu} + g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma}) + \delta_\nu^\alpha(\phi_{;\mu\beta} - \phi_{;\mu}\phi_{;\beta} + g_{\mu\beta}\phi^{;\sigma}\phi_{;\sigma}) \\ \quad - g_{\mu\nu}(\phi^{;\alpha}_{\beta} - \phi^{;\alpha}\phi_{;\beta}) + g_{\mu\beta}(\phi^{;\alpha}_{\nu} - \phi^{;\alpha}\phi_{;\nu}) \\ \tilde{R}_{\mu\nu} = R_{\mu\nu} - 2\phi_{;\mu\nu} + 2\phi_{;\mu}\phi_{;\nu} - g_{\mu\nu}\Box\phi - 2g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \\ \tilde{R} = e^{-2\phi}(R - 6\Box\phi - 6\phi_{;\sigma}\phi^{;\sigma}) \\ \phi_{;\mu\nu} = \phi_{;\mu\nu} - \tilde{\Gamma}_{\alpha\beta}^\sigma\phi_{;\sigma} = \phi_{;\mu\nu} - 2\phi_{;\mu}\phi_{;\nu} + g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \\ \tilde{G}_{\mu\nu} = G_{\mu\nu} - 2\phi_{;\mu\nu} + 2\phi_{;\mu}\phi_{;\nu} + 2g_{\mu\nu}\Box\phi + g_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \end{array} \right. \quad (28)$$

where $\phi \doteq \ln A^{1/2}$. The inverse relations are

$$\left\{ \begin{array}{l} \Gamma_{\mu\nu}^\alpha = \tilde{\Gamma}_{\mu\nu}^\alpha - \phi_{;\mu}\delta_\nu^\alpha - \phi_{;\nu}\delta_\mu^\alpha + \tilde{\phi}^{;\alpha}\tilde{g}_{\mu\nu} \\ R^\alpha_{\mu\beta\nu} = \tilde{R}^\alpha_{\mu\beta\nu} + \delta_\beta^\alpha(\tilde{\phi}_{;\mu\nu} + \phi_{;\mu}\phi_{;\nu}) - \delta_\nu^\alpha(\tilde{\phi}_{;\mu\beta} + \phi_{;\mu}\phi_{;\beta}) + \tilde{g}_{\mu\nu}(\tilde{\phi}^{;\alpha}_{\beta} - \tilde{\phi}^{;\alpha}\phi_{;\beta}) - \tilde{g}_{\mu\beta}(\tilde{\phi}^{;\alpha}_{\nu} - \tilde{\phi}^{;\alpha}\phi_{;\nu}) \\ R_{\mu\nu} = \tilde{R}_{\mu\nu} + 2\tilde{\phi}_{;\mu\nu} + 2\phi_{;\mu}\phi_{;\nu} + \tilde{g}_{\mu\nu}\tilde{\Box}\phi - 2\tilde{g}_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \\ R = e^{2\phi}(\tilde{R} + 6\tilde{\Box}\phi - 6\phi^{;\sigma}\phi_{;\sigma}) \\ \phi_{;\mu\nu} = \phi_{;\mu\nu} - \Gamma_{\alpha\beta}^\sigma\phi_{;\sigma} = \tilde{\phi}_{;\mu\nu} + 2\phi_{;\mu}\phi_{;\nu} - \tilde{g}_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \\ G_{\mu\nu} = \tilde{G}_{\mu\nu} + 2\tilde{\phi}_{;\mu\nu} + 2\phi_{;\mu}\phi_{;\nu} - 2\tilde{g}_{\mu\nu}\tilde{\Box}\phi + \tilde{g}_{\mu\nu}\phi^{;\sigma}\phi_{;\sigma} \end{array} \right. \quad (29)$$

where \Box and $\tilde{\Box}$ are the d'Alembert operators with respect to the metric $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$. The transformation between the d'Alembert operators is $\Box = e^{2\phi}\tilde{\Box} - 2\phi^{;\nu}\partial_\nu$.

Under these transformations, the action (24) can be reformulated as follows

$$\mathcal{A}_{EF}^{ST} = \int d^4x \sqrt{-\tilde{g}} \left[\Lambda \tilde{R} + \Omega(\varphi)\varphi_{;\alpha}\varphi^{;\alpha} + W(\varphi) + \mathcal{K}\tilde{\mathcal{L}}_m \right] \quad (30)$$

in which \tilde{R} is the Ricci scalar relative to the metric \tilde{g} and Λ is a generic constant. The relations between the quantities in two frames are

$$\left\{ \begin{array}{l} \Omega(\varphi)d\varphi^2 = \Lambda \left[\frac{\omega(\phi)}{F(\phi)} - \frac{3}{2} \left(\frac{d \ln F(\phi)}{d\phi} \right)^2 \right] d\phi^2 \\ W(\varphi) = \frac{\Lambda^2}{F(\phi(\varphi))^2} V(\phi(\varphi)) \\ \tilde{\mathcal{L}}_m = \frac{\Lambda^2}{F(\phi(\varphi))^2} \mathcal{L}_m \left(\frac{\Lambda \tilde{g}_{\rho\sigma}}{F(\phi(\varphi))} \right) \\ F(\phi)A(x^\lambda)^{-1} = \Lambda \end{array} \right. \quad (31)$$

The field equations for the new fields $\tilde{g}_{\mu\nu}$ and φ are

$$\left\{ \begin{array}{l} \Lambda \tilde{G}_{\mu\nu} - \frac{1}{2} W(\varphi) \tilde{g}_{\mu\nu} + \Omega(\varphi) \left[\varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} \varphi_{;\alpha} \varphi^{;\alpha} \tilde{g}_{\mu\nu} \right] = \mathcal{X} \tilde{T}_{\mu\nu}^{\varphi} \\ 2\Omega(\varphi) \tilde{\square} \varphi - \Omega_{,\varphi}(\varphi) \varphi_{;\alpha} \varphi^{;\alpha} - W_{,\varphi}(\varphi) = \mathcal{X} \tilde{\mathcal{L}}_{m,\varphi} \\ \tilde{R} = -\frac{1}{\Lambda} \left(\mathcal{X} \tilde{T}^{\varphi} + 2W(\varphi) + \Omega(\varphi) \tilde{g}^{\sigma\tau} \varphi_{;\sigma} \varphi_{;\tau} \right) \end{array} \right. \quad (32)$$

Therefore, any non-minimally coupled scalar-tensor theory, in absence of ordinary matter, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential suitably defined by (31). The converse is also true: for a given $F(\phi)$, such that the relations (31) hold, we can transform a standard Einstein theory plus scalar field into a non-minimally coupled scalar-tensor theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get solutions for the scalar-tensor theories, assigned by the coupling $F(\phi)$, via the conformal transformation (27) with the constraints given by (31). Following the standard terminology, the “Einstein frame” is the framework of the Einstein theory with the minimal coupling and the “Jordan frame” is the framework of the non-minimally coupled theory [34, 35]. These considerations will be extremely useful for the discussion below.

III. SPHERICAL SYMMETRY

Spherical symmetry is the first step necessary to develop the Newtonian and the post-Newtonian limits of any alternative theory of gravity. Here, we will discuss the basic features related to such a symmetry starting from GR and then we develop a perturbation approach for spherically symmetric solutions of $f(R)$ -gravity. This machinery will be extremely useful to develop the FOG weak-field limit.

A. Generalities on spherical symmetry

Since we are interested to understand the modifications of GR predictions when one takes into account concentrations of matter in the space, it is fundamental to discuss properties of the metric $g_{\mu\nu}$. As a first step, we will study the gravitational potential generated by spherically symmetric distributions of matter. The most general spherically symmetric metric² can be written as

$$ds^2 = g_1(t, |\mathbf{x}|) dt^2 + g_2(t, |\mathbf{x}|) dt \mathbf{x} \cdot d\mathbf{x} + g_3(t, |\mathbf{x}|) (\mathbf{x} \cdot d\mathbf{x})^2 + g_4(t, |\mathbf{x}|) d|\mathbf{x}|^2 \quad (33)$$

where g_i are functions of the spatial distance $|\mathbf{x}|$ and of the time t . The set of coordinates is $x^\mu = (t, x^1, x^2, x^3)$. The scalar product is defined as $\mathbf{x} \cdot d\mathbf{x} = x^1 dx^1 + x^2 dx^2 + x^3 dx^3$. By the spherically symmetric form of (33), it is convenient to replace \mathbf{x} with spherical coordinates r, θ, ϕ defined as

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \theta \quad (34)$$

The line element (33) then becomes

$$ds^2 = g_1(t, r) dt^2 + r g_2(t, r) dt dr + r^2 g_3(t, r) dr^2 + g_4(t, r) (dr^2 + r^2 d\Omega) \quad (35)$$

where $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$ is the solid angle. We are free to reset our clocks by defining the time coordinate

$$t = t' + \epsilon(t', r) \quad (36)$$

² The metric is spherically symmetric if it depends only on \mathbf{x} and $d\mathbf{x}$ only through the rotational invariants $d\mathbf{x}^2$, $\mathbf{x} \cdot d\mathbf{x}$ and \mathbf{x}^2 .

with $\epsilon(t', r)$ an arbitrary function of t' and r . This allows us to eliminate the off-diagonal element g_{tr} in the metric (35) by setting

$$\frac{d\epsilon(t', r)}{dr} = -\frac{rg_2(t', r)}{2g_1(t', r)} \quad (37)$$

The metric (35) becomes

$$ds^2 = g_1(t', r) \left[1 + \frac{d\epsilon(t', r)}{dt'} \right]^2 dt'^2 + \left[r^2 g_3(t', r) - \frac{r^2 g_2(t', r)^2}{4g_1(t', r)} + g_4(t', r) \right] dr^2 + g_4(t', r) r^2 d\Omega \quad (38)$$

where introducing a new metric coefficients $g_{tt}(t', r)$, $g_{rr}(t', r)$ and $g_{\Omega\Omega}(t', r)$, we can recast Eq. (38) as follows

$$ds^2 = g_{tt}(t', r) dt'^2 - g_{rr}(t', r) dr^2 - g_{\Omega\Omega}(t', r) d\Omega \quad (39)$$

Introducing a new radial coordinate (r') by considering the further transformation

$$r' = (const) e^{\int dr \sqrt{\frac{g_{rr}(t', r)}{g_{\Omega\Omega}(t', r)}}} \quad (40)$$

it is possible to recast Eq.(39) into the isotropic form (isotropic coordinates)

$$ds^2 = g_{tt}(t', r') dt'^2 - g_{ij}(t', r') dx^i dx^j \quad (41)$$

and then it is possible to choose $g_{\Omega\Omega}(t', r) = r''^2$ (this condition allows to obtain the standard definition of the circumference with radius r'') and to have the metric (39) in the standard form (standard coordinates)

$$ds^2 = g_{tt}(t', r'') dt'^2 - g_{rr}(t', r'') dr''^2 - r''^2 d\Omega \quad (42)$$

Obviously the functions $g_{tt}(t', r'')$ and $g_{rr}(t', r'')$ are not the same of (39). If we suppose $g_{ij}(t', r') = Y(t', r') \delta_{ij}$, it is worth noticing that it is possible to pass from (41) to (42) by the coordinate transformation

$$r' = r'(r'') = (const) e^{\int dr'' \frac{\sqrt{Y(r'')}}{r''}} \quad (43)$$

We can, then, state that the expressions (39), (41) and (42) are equivalent to the metric (33) and we can consider them without loss of generality as the most general definitions of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion. The choice of the metric form is only a practical issue useful to develop calculations.

B. The Birkhoff theorem in General Relativity

The Birkhoff theorem is an important result of GR which essentially states that stationary solutions are also static and stable. Specifically, the theorem holds for spherically symmetric solutions.

Let us start our considerations by rewriting the metric (42) as follows

$$ds^2 = e^{\nu(t, r)} dt^2 - e^{\mu(t, r)} dr^2 - r^2 d\Omega \quad (44)$$

where we redefined the radial coordinate. The only non-vanishing components of metric tensor $g_{\mu\nu}$ are

$$g_{tt} = e^{\nu(t, r)}, \quad g_{rr} = -e^{\mu(t, r)}, \quad g_{\theta\theta} = -r^2, \quad g_{\phi\phi} = -r^2 \sin^2 \theta \quad (45)$$

with functions $\mu(t, r)$ and $\nu(t, r)$ that are to be determined by solving the field equations in GR (9). Since $g_{\mu\nu}$ is diagonal, it is easy to write all the non-vanishing inverse components:

$$g^{tt} = e^{-\nu(t, r)}, \quad g^{rr} = -e^{-\mu(t, r)}, \quad g^{\theta\theta} = -r^{-2}, \quad g^{\phi\phi} = -r^{-2} \sin^2 \theta \quad (46)$$

Furthermore, the determinant of the metric tensor is

$$g = -e^{\mu(t, r) + \nu(t, r)} r^4 \sin^2 \theta \quad (47)$$

so the invariant volume element is

$$\sqrt{-g} dr d\theta d\phi = r^2 e^{-\frac{\mu(t, r) + \nu(t, r)}{2}} \sin \theta dr d\theta d\phi \quad (48)$$

The only non-vanishing components of Christoffel symbols (3) are

$$\left\{ \begin{array}{lll} \Gamma_{tt}^t = \frac{\dot{\nu}(t, r)}{2} & \Gamma_{rr}^r = \frac{\mu'(t, r)}{2} & \Gamma_{tt}^r = \frac{\nu'(t, r)}{2} e^{\nu(t, r) - \mu(t, r)} \\ \Gamma_{rr}^t = \frac{\mu'(t, r)}{2} e^{\mu(t, r) - \nu(t, r)} & \Gamma_{tr}^t = \frac{\nu'(t, r)}{2} & \Gamma_{tr}^r = \frac{\nu(t, r)}{2} \\ \Gamma_{\theta\theta}^r = -r e^{-\mu(t, r)} & \Gamma_{r\theta}^\theta = \Gamma_{r\phi}^\phi = \frac{1}{r} & \Gamma_{\phi\phi}^r = -r e^{-\mu(t, r)} \sin^2 \theta \\ \Gamma_{\phi\phi}^\theta = -\sin \theta \cos \theta & \Gamma_{\theta\phi}^\phi = \cot \theta & \end{array} \right. \quad (49)$$

and the field equations (9) become

$$\left\{ \begin{array}{l} \frac{1}{r^2} - e^{-\mu(t, r)} \left[\frac{1}{r^2} - \frac{\mu'(t, r)}{r} \right] = \mathcal{X} T_t^t \\ \frac{\dot{\mu}(t, r)}{r} e^{-\mu(t, r)} = \mathcal{X} T_t^r \\ \frac{1}{r^2} - e^{-\mu(t, r)} \left[\frac{\nu'(t, r)}{r} + \frac{1}{r^2} \right] = \mathcal{X} T_r^r \\ \frac{e^{-\nu(t, r)}}{2} \left[\ddot{\mu}(t, r) + \frac{\dot{\mu}^2(t, r)}{2} - \frac{\dot{\mu}(t, r) \dot{\nu}(t, r)}{2} \right] \\ - \frac{e^{-\mu(t, r)}}{2} \left[\nu''(t, r) + \frac{\nu'^2(t, r)}{2} + \frac{\nu'(t, r) - \mu'(t, r)}{r} - \frac{\nu'(t, r) \mu'(t, r)}{2} \right] = \mathcal{X} T_\theta^\theta = \mathcal{X} T_\phi^\phi \end{array} \right. \quad (50)$$

If we suppose a stress-energy tensor (7) induced by a point-like source with mass M

$$\left\{ \begin{array}{l} T_{\mu\nu} = \rho(\mathbf{x}) u_\mu u_\nu \\ T = \rho(\mathbf{x}) \end{array} \right. \quad (51)$$

where $\rho = M \delta(\mathbf{x})$, we obtain the *Schwarzschild solution* in standard coordinates³

$$ds^2 = \left[1 - \frac{r_g}{r} \right] dt^2 - \frac{dr'^2}{1 - \frac{r_g}{r}} - r'^2 d\Omega \quad (52)$$

³ $\delta(\mathbf{x})$ is the 3-dimensional Dirac δ -function.

where $r_g = 2GM$ is the *Schwarzschild radius*.

Metric (52) determines completely the gravitational field in the vacuum generated by a spherical matter density distribution. Furthermore the Schwarzschild solution is valid also when we consider a moving source with a spherical distribution. The spatial metric is determined by expression of spatial distance element

$$dl^2 = \frac{dr''^2}{1 - \frac{r_g}{r''}} + r''^2 d\Omega \quad (53)$$

We have to note that, while the length of circumference with "radius" r'' is the usual one $2\pi r''$, the distance between two points on the same radius is given by the integral

$$\int_{r_1''}^{r_2''} \frac{dr''}{\sqrt{1 - \frac{r_g}{r''}}} > r_2'' - r_1'' \quad (54)$$

this means that the space is curved. Besides we note that $g_{tt} \leq 1$, then, by the relation between the time coordinate t and the proper time τ ($d\tau = \sqrt{g_{tt}} dt$), we get the condition

$$d\tau \leq dt \quad (55)$$

At infinity, the coordinate time coincides with the physical time. We can state that when we are at a finite distance from the mass, there is a slowdown of the time with respect to the time measured at infinity.

In presence of matter the situation is the following. From the first equation in the (50), when $r \rightarrow 0$, $\mu(t, r)$ has to vanish as r^2 ; otherwise T^t_t could have a singular point in the origin. By formally integrating the equation with the condition $\mu(t, r)|_{r=0} = 0$, one gets

$$\mu(t, r) = -\ln \left[1 - \frac{\mathcal{X}}{r} \int_0^r T^t_t \hat{r}^2 d\hat{r} \right] \quad (56)$$

Beside the point-like case, it is easy to demonstrate that the proprieties (54), (55) and $\mu(t, r) + \nu(t, r) \leq 0$ hold also in more general matter distributions with spherical symmetry [25]. If the gravitational field is generated by a spherical body with "radius" ξ , we have $T^t_t = 0$ outside the body ($r > \xi$) and we can write

$$\mu_\xi(t, r) = -\ln \left[1 - \frac{\mathcal{X}}{r} \int_0^\xi T^t_t \hat{r}^2 d\hat{r} \right] \quad (57)$$

obtaining the analogous expression of (52) in the matter

$$ds^2 = \left[1 - \frac{r_g(r'')}{r''} \right] dt^2 - \frac{dr''^2}{1 - \frac{r_g(r'')}{r''}} - r''^2 d\Omega \quad (58)$$

where we have introduced the Schwarzschild radius related to the quantity of matter included in the sphere with radius r'' :

$$r_g(r'') = \mathcal{X} \int_0^{r''} T^t_t \hat{r}^2 d\hat{r} \quad (59)$$

Obviously when the distance is bigger than the radius of the body, the metric (58) is equal to (52).

If we consider the transformation (43), which in the case of Schwarzschild solution is

$$r' = \frac{2r'' - r_g + 2\sqrt{r''^2 - r_g r''}}{4} \quad (60)$$

it is possible to obtain the Schwarzschild solution (52) in isotropic coordinates

$$ds^2 = \left[\frac{1 - \frac{r_g}{4r'}}{1 + \frac{r_g}{4r'}} \right]^2 dt^2 - \left[1 + \frac{r_g}{4r'} \right]^4 (dr'^2 + r'^2 d\Omega) \quad (61)$$

In both cases, solutions (52) and (58) obviously satisfy the trace of field equations: $R = -\mathcal{X}T$. Since in the vacuum the trace of matter tensor is vanishing (except in the origin, where the trace is proportional to $\delta(\mathbf{x})$) we can state that the Schwarzschild solution is "Ricci flat": $R = 0$.

If we add in the Hilbert - Einstein Lagrangian (12) a term like $(-2\sqrt{-g}\Lambda)$, with Λ a generic constant, the field equations (9) are modified as follows

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{X} T_{\mu\nu} \quad (62)$$

In the case of a point-like source, we find the *Schwarzschild - de Sitter solution*

$$ds^2 = \left[1 - \frac{r_g}{r''} + \frac{\Lambda}{3} r''^2 \right] dt^2 - \frac{dr''^2}{1 - \frac{r_g}{r''} + \frac{\Lambda}{3} r''^2} - r''^2 d\Omega \quad (63)$$

The trace of (62) is

$$R = 4\Lambda - \mathcal{X}T \quad (64)$$

from which we note that this solution does not admit solution in vacuum, since also in absence of ordinary matter ($T_{\mu\nu} = 0$) we have a non-vanishing scalar curvature. The contribution is given by the cosmological constant Λ . Also in this case, analogous considerations as in (58) hold.

Finally let us consider as source a radial and static electric field $\mathbf{E} = Q\mathbf{x}/|\mathbf{x}|^3$. We know that the Lagrangian of electromagnetic field is $-\frac{1}{4\pi}F_{\alpha\beta}F^{\alpha\beta}$ where $F_{\alpha\beta}$ is the electromagnetic tensor. Then, the Hilbert - Einstein Lagrangian is

$$\mathcal{L}_{HE} = \sqrt{-g}(R - \frac{1}{4\pi} F_{\alpha\beta}F^{\alpha\beta}) \quad (65)$$

and the Einstein equation (9) becomes

$$G_{\mu\nu} = -\frac{1}{8\pi} (g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - 4F_{\mu\alpha}F^{\alpha}_{\nu}) \quad (66)$$

The solution for a spherically symmetric system is the *Reissner - Nordstrom solution*

$$ds^2 = \left[1 - \frac{r_g}{r''} + \frac{Q^2}{r''^2} \right] dt^2 - \frac{dr''^2}{1 - \frac{r_g}{r''} + \frac{Q^2}{r''^2}} - r''^2 d\Omega \quad (67)$$

In all the above cases, the Birkhoff theorem holds: *The metric tensor generated in vacuum by a matter density distribution with a spherical symmetry is time-independent.* Also a time-dependent source with a spherical symmetry produces a static metric. The curvature of spacetime in the matter, at a distance r from the origin, is proportional only to the matter inside the sphere of radius r . This conclusion is compatible with the Gauss theorem of Classical Mechanics.

C. The Schwarzschild solution and the Eddington parameterization

The Schwarzschild radius r_g in the solution (61) is a scale-length induced by theory and represents a natural parameter to study the gravitational interaction with respect to the generating source. At radial distance r' where $r_g/r' \ll 1$, solution (61) becomes

$$ds^2 \simeq \left[1 - \frac{r_g}{r'} + \frac{1}{2} \left(\frac{r_g}{r'} \right)^2 + \dots \right] dt^2 - \left[1 + \frac{r_g}{r'} + \dots \right] \left[dr'^2 + r'^2 d\Omega \right] \quad (68)$$

that is the standard situation presented by any self-gravitating system far from its critical radius.

Since we are interested to investigate the deviations, induced by FOG, from the behavior (68). it is useful to introduce the method taking into account such deviations with respect to GR. The standard approach is the Parameterized-Post-Newtonian (PPN) expansion of the Schwarzschild metric (61). Eddington parameterized deviations with respect to GR considering a Taylor expansion in term of r_g/r' and assumed that, in Solar System, the limit $r_g/r' \ll 1$ holds [21]. The resulting metric is

$$ds^2 \simeq \left[1 - \alpha \frac{r_g}{r'} + \frac{\beta}{2} \left(\frac{r_g}{r'} \right)^2 + \dots \right] dt^2 - \left[1 + \gamma \frac{r_g}{r'} + \dots \right] \left[dr'^2 + r'^2 d\Omega \right] \quad (69)$$

where α , β and γ are dimensionless parameters (Eddington parameters) which parameterize deviations with respect to GR. The reason to carry out this expansion up to the order $(r_g/r')^2$ in g_{tt} and only to the order (r_g/r') in g_{ij} is that, in applications to Celestial Mechanics, g_{ij} always appears multiplied by an extra factor $v^2 \sim M/r'$. It is evident that the standard GR solution for a spherically symmetric gravitational system in vacuum, is obtained for $\alpha = \beta = \gamma = 1$ giving again the "approximated" Schwarzschild solution (68). Actually, the parameter α can be settled to the unity due to the mass definition of the system itself [21]. As a consequence, the expanded metric (69) can be recast in the form:

$$ds^2 \simeq \left[1 - \frac{r_g}{r''} + \frac{\beta - \gamma}{2} \left(\frac{r_g}{r''} \right)^2 + \dots \right] dt^2 - \left[1 + \gamma \frac{r_g}{r''} + \dots \right] dr''^2 - r''^2 d\Omega \quad (70)$$

where we have restored the standard spherical coordinates by means of the transformation $r'' = r' \left[1 + \frac{r_g}{4r'} \right]^2$. The parameters β , γ have a physical interpretation. The parameter γ measures the amount of curvature of space generated by a body of mass M at radius r' . In fact, the spatial components of the Riemann curvature tensor are, at post-Newtonian order,

$$R_{ijkl} = \frac{3}{2} \gamma \frac{r_g}{r'^3} N_{ijkl} \quad (71)$$

independently of the gauge choice, where N_{ijkl} represents the geometric tensor properties (e.g. symmetries of the Riemann tensor and so on). On the other hand, the parameter β measures the amount of non-linearity ($\sim (r_g/r')^2$) in the g_{tt} component of the metric. However, this statement is valid only in the standard post-Newtonian gauge.

These considerations can be developed for any relativistic theory of gravity but, as we shall see below, the above results, valid in GR, cannot be simply extrapolated to any modified theory of gravity since misleading conclusions could be achieved. For example, in literature, there are some papers stating that $f(R) \neq R$ are not viable models in Solar System since, by recasting the $f(R)$ -gravity as the O'Hanlon model, $\gamma 1/2$ [47] in evident contrast with GR measurement giving $\gamma \simeq 1$. On the other hand, assuming $f(R) = R^{1+\epsilon}$ with $\epsilon \rightarrow 0$, the standard GR would be hard to be recovered in the weak field limit and the Cauchy boundary conditions [48, 49] would be highly violated if the theory results switched from $\gamma = 1$, for $\epsilon = 0$, to $\gamma = 1/2$, for $\epsilon \neq 0$. As shown in [55], the shortcoming can be solved by considering separately the weak field limits of $f(R)$ and O'Hanlon theory which have to be confronted "after" the PPN-approximation. The conceptual reason is clear: the invariance properties of the theories are lost in the approximation process so the confront has to be performed assuming the same gauge and formalism. In other words, different theories have to be carefully confronted at general level and at approximated level without incautious extrapolations. This argument will be considered in detail below.

D. Perturbing the spherically symmetric solutions of $f(R)$ -gravity

Let us focus now on the differences of perturbing a spherically symmetric solution coming from a generic $f(R)$ -gravity model with respect to the standard spherically symmetric solutions coming from the GR case $f(R) = R$. We

neglect, for the moment, the contributions of Ricci and Riemann square [36] which will be considered below. The search for solutions in FOG can be faced by means of a perturbation theory.

The general approach is to start from analytical $f(R)$ -function assuming that the background model slightly deviates from GR. Such a method can provide interesting results on astrophysical scales where spherically symmetric solutions, characterized by small values of the scalar curvature, can be taken into account. Below, we will consider the perturbing approach assuming that the background metric matches, at zero order, the GR solutions.

The field equations (21) and (22) in $f(R)$ -gravity, if we develop also the covariant derivatives, become

$$\begin{cases} H_{\mu\nu} = f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\ H = f' R - 2f + \mathcal{H} = \mathcal{X} T \end{cases} \quad (72)$$

where the two quantities $\mathcal{H}_{\mu\nu}$ and \mathcal{H} read

$$\begin{cases} \mathcal{H}_{\mu\nu} = -f'' \left\{ R_{,\mu\nu} - \Gamma_{\mu\nu}^{\sigma} R_{,\sigma} - g_{\mu\nu} \left[\left(g^{\sigma\tau}{}_{,\sigma} + g^{\sigma\tau} \ln \sqrt{-g}{}_{,\sigma} \right) R_{,\tau} + g^{\sigma\tau} R_{,\sigma\tau} \right] \right\} - f''' \left(R_{,\mu} R_{,\nu} - g_{\mu\nu} g^{\sigma\tau} R_{,\sigma} R_{,\tau} \right) \\ \mathcal{H} = 3f'' \left[\left(g^{\sigma\tau}{}_{,\sigma} + g^{\sigma\tau} \ln \sqrt{-g}{}_{,\sigma} \right) R_{,\tau} + g^{\sigma\tau} R_{,\sigma\tau} \right] + 3f''' g^{\sigma\tau} R_{,\sigma} R_{,\tau} \end{cases} \quad (73)$$

where f^i is the i -th derivative of f with respect to Ricci scalar R .

In general, searching for solutions by a perturbation technique means to perturb the metric

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + g_{\mu\nu}^{(1)} \quad (74)$$

This implies that the field equations (72) split, up to first order, in two parts. Besides, a perturbation on the metric acts also on the Ricci scalar R defined in Eq.(5)

$$R \sim R^{(0)} + R^{(1)} \quad (75)$$

and then we can Taylor expand the analytic function $f(R)$ about the background value of R , *i.e.*:

$$\begin{cases} f = \sum_n \frac{f^{(n)}(R^{(0)})}{n!} \left[R - R^{(0)} \right]^n = \sum_n \frac{f^{(n)}(0)}{n!} R^{(1)n} \sim f^{(0)} + f'^{(0)} R^{(1)} + \frac{f''^{(0)}}{2} R^{(1)2} + \frac{f'''^{(0)}}{6} R^{(1)3} + \frac{f^{IV(0)}}{24} R^{(1)4} \\ f' = \sum_n \frac{f^{(n+1)}(R^{(0)})}{n!} \left[R - R^{(0)} \right]^n \sim f'^{(0)} + f''^{(0)} R^{(1)} + \frac{f'''^{(0)}}{2} R^{(1)2} + \frac{f^{IV(0)}}{6} R^{(1)3} = \frac{df}{dR^{(1)}} \\ f'' = \sum_n \frac{f^{(n+2)}(R^{(0)})}{n!} \left[R - R^{(0)} \right]^n \sim f''^{(0)} + f'''^{(0)} R^{(1)} + \frac{f^{IV(0)}}{2} R^{(1)2} = \frac{df'}{dR^{(1)}} \\ f''' = \sum_n \frac{f^{(n+3)}(R^{(0)})}{n!} \left[R - R^{(0)} \right]^n \sim f'''^{(0)} + f^{IV(0)} R^{(1)} = \frac{df''}{dR^{(1)}} \end{cases} \quad (76)$$

The zero order of field equations (72) reads

$$f'^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} f^{(0)} + \mathcal{H}_{\mu\nu}^{(0)} = \mathcal{X} T_{\mu\nu}^{(0)} \quad (77)$$

where

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(0)} = & -f''^{(0)} \left\{ R_{,\mu\nu}^{(0)} - \Gamma_{\mu\nu}^{(0)\rho} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left(g^{(0)\rho\sigma}{}_{,\rho} R_{,\sigma}^{(0)} + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \ln \sqrt{-g}{}_{,\rho} R_{,\sigma}^{(0)} \right) \right\} \\ & - f'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \right\} \end{aligned} \quad (78)$$

At first order, one has :

$$f'^{(0)} \left\{ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} \right\} + f''^{(0)} R^{(1)} R_{\mu\nu}^{(0)} - \frac{1}{2} f^{(0)} g_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(1)} = \mathcal{X} T_{\mu\nu}^{(1)} \quad (79)$$

with

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(1)} = & -f''^{(0)} \left\{ R_{\mu\nu}^{(1)} - \Gamma_{\mu\nu}^{(0)\rho} R_{,\rho}^{(1)} - \Gamma_{\mu\nu}^{(1)\rho} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left[g^{(0)\rho\sigma}{}_{,\rho} R_{,\sigma}^{(1)} + g^{(1)\rho\sigma}{}_{,\rho} R_{,\sigma}^{(0)} \right. \right. \\ & + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(1)} + g^{(1)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \left(\ln \sqrt{-g_{,\rho}^{(0)}} R_{,\sigma}^{(1)} + \ln \sqrt{-g_{,\rho}^{(1)}} R_{,\sigma}^{(0)} \right) \\ & + g^{(1)\rho\sigma} \ln \sqrt{-g_{,\rho}^{(0)}} R_{,\sigma}^{(0)} \left. \right] - g_{\mu\nu}^{(1)} \left(g^{(0)\rho\sigma}{}_{,\rho} R_{,\sigma}^{(0)} + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} \right. \\ & + g^{(0)\rho\sigma} \ln \sqrt{-g_{,\rho}^{(0)}} R_{,\sigma}^{(0)} \left. \right) \left. \right\} - f'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(1)} + R_{,\mu}^{(1)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} \left(R_{,\rho}^{(0)} R_{,\sigma}^{(1)} \right. \right. \\ & + R_{,\rho}^{(1)} R_{,\sigma}^{(0)} \left. \right) - g_{\mu\nu}^{(0)} g^{(1)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} - g_{\mu\nu}^{(1)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \left. \right\} - f'''^{(0)} R^{(1)} \left\{ R_{,\mu\nu}^{(0)} \right. \\ & - \Gamma_{\mu\nu}^{(0)\rho} R_{,\rho}^{(0)} - g_{\mu\nu}^{(0)} \left(g^{(0)\rho\sigma}{}_{,\rho} R_{,\sigma}^{(0)} + g^{(0)\rho\sigma} R_{,\rho\sigma}^{(0)} + g^{(0)\rho\sigma} \ln \sqrt{-g_{,\rho}^{(0)}} R_{,\sigma}^{(0)} \right) \left. \right\} \\ & - f^{IV(0)} R^{(1)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \right\} \end{aligned} \quad (80)$$

$$- f^{IV(0)} R^{(1)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\rho\sigma} R_{,\rho}^{(0)} R_{,\sigma}^{(0)} \right\} \quad (81)$$

A part the analyticity, no hypothesis has been invoked here on the form of the function $f(R)$. As a matter of fact, $f(R)$ can be completely general. At this level, to solve the problem, it is required the zero order solution of (77) which, in general, *could not be a GR solution*. This problem can be overcome assuming the same order of perturbation on the $f(R)$, that is:

$$f(R) = R + \mathcal{F}(R) \quad (82)$$

where $\mathcal{F}(R)$ is a generic function of the Ricci scalar (this means to consider $\mathcal{F}(R) \ll R$). Then we have

$$f = R^{(0)} + R^{(1)} + \mathcal{F}^{(0)}, \quad f' = 1 + \mathcal{F}'^{(0)}, \quad f'' = \mathcal{F}''^{(0)}, \quad f''' = \mathcal{F}'''^{(0)} \quad (83)$$

However the condition $\mathcal{F}(R) \ll R$ has to imply the validity of the linear approximation $\frac{f''(R^{(0)})R^{(1)}}{f'(R^{(0)})} \ll 1$. This is demonstrated by assuming $f' = 1 + \mathcal{F}'$ and $f'' = \mathcal{F}''$. Immediately, we obtain that the condition is fulfilled for

$$\frac{\mathcal{F}''(R^{(0)})R^{(1)}}{1 + \mathcal{F}'(R^{(0)})} \ll 1 \quad (84)$$

For example, given a Lagrangian of the form $f(R) = R + \frac{R_0}{R}$, Eq.(84) becomes

$$\frac{2R_0 R^{(1)}}{R^{(0)}(R^{(0)2} - R_0)} \ll 1 \quad (85)$$

while, for $f(R) = R + \alpha R^2$, Eq.(84) becomes

$$\frac{2\alpha R^{(1)}}{1 + 2\alpha R^{(0)}} \ll 1 \quad (86)$$

This means that the validity of the approximation strictly depends on the form of the models and the value of the parameters, in the previous cases, R_0 and α . For the considerations below, we will assume that condition (84) always holds.

Eqs.(77) reduce to the GR form

$$R_{\mu\nu}^{(0)} - \frac{1}{2}R^{(0)}g_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} = \mathcal{X}T_{\mu\nu}^{(0)} \quad (87)$$

On the other hand, Eqs. (79) reduce to

$$R_{\mu\nu}^{(1)} - \frac{1}{2}g_{\mu\nu}^{(0)}R^{(1)} - \frac{1}{2}g_{\mu\nu}^{(1)}R^{(0)} - \frac{1}{2}g_{\mu\nu}^{(0)}\mathcal{F}^{(0)} + \mathcal{F}'^{(0)}R_{\mu\nu}^{(0)} + \mathcal{H}_{\mu\nu}^{(1)} = \mathcal{X}T_{\mu\nu}^{(1)} \quad (88)$$

where

$$\begin{aligned} \mathcal{H}_{\mu\nu}^{(1)} = & -\mathcal{F}'''^{(0)} \left\{ R_{,\mu}^{(0)} R_{,\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)\sigma\tau} R_{,\sigma}^{(0)} R_{,\tau}^{(0)} \right\} \\ & -\mathcal{F}''^{(0)} \left\{ R_{,\mu\nu}^{(0)} - \Gamma_{\mu\nu}^{(0)\sigma} R_{,\sigma}^{(0)} - g_{\mu\nu}^{(0)} \left(g^{(0)\sigma\tau}{}_{,\sigma} R_{,\tau}^{(0)} + g^{(0)\sigma\tau} R_{,\sigma\tau}^{(0)} + g^{(0)\sigma\tau} \ln \sqrt{-g_{,\sigma}^{(0)}} R_{,\tau}^{(0)} \right) \right\} \end{aligned} \quad (89)$$

The new system of field equations is evidently simpler than the starting one and once the zero order solution is obtained, the solutions at the first order correction can be easily achieved.

By assuming the time-independent metric (42)

$$ds^2 = a(r)dt^2 - b(r)dr^2 - r^2 d\Omega^2 \quad (90)$$

it is straightforward to obtain new spherically symmetric solutions by substituting into the above perturbed field equations. In Tables I and II, a list of solutions, obtained with this perturbation method, is given considering different classes of $f(R)$ -models. Some remarks on these solutions are in order at this point. In the case of $f(R)$ -models which are evidently corrections to the Hilbert-Einstein Lagrangian as $\Lambda + R + \epsilon R \ln R$ and $R + \epsilon R^n$, with $\epsilon \ll 1$, one obtains exact solutions for the gravitational potentials $a(r)$ and $b(r)$ related by $a(r) = b(r)^{-1}$. The first order expansion is straightforward as in GR. If the functions $a(r)$ and $b(r)$ are not related, for $f(R) = \Lambda + R + \epsilon R \ln R$, the first order system is directly solved without any prescription on the perturbation functions $x(r)$ and $y(r)$ indicated in the Tables. This is not the case for the models $f(R) = R + \epsilon R^n$ since, in this case, one can obtain an explicit constraint on the perturbation function. In such a case, no corrections are found with respect to the standard solution. The models $f(R) = R^n$ and $f(R) = \frac{R}{(R_0+R)}$ show similar behaviors. The case $f(R) = R^2$ is peculiar and it has to be dealt independently. For details, see [45].

IV. GENERAL REMARKS ON NEWTONIAN AND POST-NEWTONIAN APPROXIMATIONS

At this point, it is worth discussing some general issues on the Newtonian and post-Newtonian limits. Basically there are some general features one has to take into account when approaching these limits, whatever the underlying theory of gravitation is. In fact here we are not interested in entering the theoretical discussion on how to formulate a mathematically well founded Newtonian and post-Newtonian limits of general relativistic field theories, nevertheless we point the interested reader to [37–43]. In this section, we provide the explicit form of the various quantities needed to compute the approximations in the field equations in GR theory and any metric theory of gravity. We only mention that there is also a discussion on alternative ways to define the Newtonian and post-Newtonian limits of FOG in the recent literature, see for example [44]. In this work, the Newtonian and post-Newtonian limits are identified by the maximally symmetric solutions, which are not necessarily the Minkowski spacetime in $f(R)$ -gravity which could be singular.

Let us start with some very general considerations on the Newtonian gravitating systems. If one takes into account a system of gravitationally interacting particles of mass \bar{M} , the kinetic energy $\frac{1}{2}\bar{M}\bar{v}^2$ will be, roughly, of the same order of magnitude as the typical potential energy $U = G\bar{M}^2/\bar{r}$, with \bar{M} , \bar{r} , and \bar{v} the typical average values of masses, separations, and velocities of these particles. As a consequence:

| | |
|-----------------------|---|
| f - theory: | $\Lambda + R + \epsilon R \ln R$ |
| spherical potentials: | $a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} - \frac{\Lambda r^2}{6} + \delta x(r)$ |
| solutions: | $x(r) = \frac{k_2}{r} + \frac{\epsilon \Lambda [\ln(-2\Lambda) - 1] r^2}{6\delta}$ |
| first order metric: | $a(r) = 1 - \frac{\Lambda r^2}{6} + \delta x(r), \quad b(r) = \frac{1}{1 - \frac{\Lambda r^2}{6}} + \delta y(r)$ |
| solutions: | $\left\{ \begin{aligned} x(r) &= (\Lambda r^2 - 6) \left\{ k_1 + \int \frac{dr}{36r\delta(\Lambda r^2 - 6)} \left[4\delta(2\Lambda^2 r^4 - 15\Lambda r^2 + 18)y(r) + r\{36r\epsilon\Lambda[\log(-2\Lambda) - 1] \right. \right. \\ &\quad \left. \left. + \delta(\Lambda r^2 - 6)^2 y'(r) \} \right] \right\} \\ y(r) &= \frac{k_2\delta - 6r^3\epsilon\Lambda[\ln(-2\Lambda) - 1]}{r\delta(r^2\Lambda - 6)^2} \end{aligned} \right\}$ |
| f - theory: | $R + \epsilon R^n$ |
| spherical potentials: | $a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \delta x(r)$ |
| solutions: | $x(r) = \frac{k_2}{r}$ |
| first order metric: | $a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$ |
| solutions: | $x(r) = k_1 + k_2 r, \quad y(r) = k_3$ |
| f - theory: | $R/(R_0 + R)$ |
| first order metric: | $a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$ |
| solutions: | $\left\{ \begin{aligned} x(r) &= -\frac{4e^{-\frac{R_0^{1/2}r}{\sqrt{6}}}}{R_0} k_1 - \frac{2\sqrt{6}e^{-\frac{R_0^{1/2}r}{\sqrt{6}}}}{R_0^{3/2}} k_2 + k_3 r \\ y(r) &= -\frac{2e^{-\frac{R_0^{1/2}r}{\sqrt{6}}}}{3b^{3/2}} (6R_0^{1/2} + \sqrt{6}R_0 r) k_1 - \frac{2e^{-\frac{R_0^{1/2}r}{\sqrt{6}}}}{R_0^{3/2}} (\sqrt{6} - R_0^{1/2} r) k_2 \end{aligned} \right\}$ |

TABLE I: A list of exact solutions obtained *via* the perturbation approach for some classes of $f(R)$ -models; k_i are integration constants; the potentials $a(r)$ and $b(r)$ are defined by the metric (90).

$$\bar{v}^2 \sim \frac{G\bar{M}}{\bar{r}} \quad (91)$$

(for instance, a test particle in a circular orbit of radius r about a central mass M has velocity v , given in Newtonian Mechanics, by the exact formula $v^2 = GM/r$.)

The post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher

| | |
|-----------------------|--|
| f - theory: | R^n |
| spherical potentials: | $a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \frac{R_0 r^2}{12} + \delta x(r)$ |
| solutions: | $\left\{ \begin{array}{l} n = 2, \quad R_0 \neq 0 \text{ and } x(r) = \frac{3k_2 - k_3}{3r} + \frac{k_3 r^2}{12} + \frac{k_4}{r} \int dr r^2 \left\{ \int dr \frac{\exp \left[\frac{R_0 r_0^2 \ln(r-r_0)}{8+3R_0 r_0^2} \right]}{r^5} \right\} \\ \quad \text{with } r_0 \text{ satisfying the condition } 6k_1 + 8r_0 + R_0 r_0^3 = 0 \\ n \geq 2, \quad \text{System solved only whit } R_0 = 0 \text{ and no prescriptions on } x(r) \end{array} \right\}$ |
| first order metric: | $a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$ |
| solutions: | $\left\{ \begin{array}{l} n = 2 \quad y(r) = -\frac{R_0 r^3}{6} - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1, \\ \quad \text{with } R(r) = \delta R_0 \\ n \neq 2 \quad y(r) = -\frac{1}{2} \int dr r^2 R(r) - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1 \\ \quad \text{with } R(r) \text{ whatever} \end{array} \right\}$ |
| first order metric: | $a(r) = 1 - \frac{r_g}{r} + \delta x(r), \quad b(r) = \frac{1}{1 - \frac{r_g}{r}} + \delta y(r)$ |
| solutions: | $\left\{ \begin{array}{l} n = 2 \quad y(r) = \frac{r k_1}{3r_g^2 - 7r_g r + 4r^2} + \frac{r^2 k_2}{3(3r_g^2 - 7r_g r + 4r^2)} + \frac{r_g r^2 x(r) + 2(r_g r^3 - r^4) x'(r)}{(3r_g - 4r)(r_g - r)^2} \\ n \neq 2 \quad \text{whatever functions } x(r), y(r) \text{ and } R(r) \end{array} \right\}$ |

TABLE II: A list of exact solutions obtained *via* the perturbation approach for some classes of $f(R)$ -models; k_i are integration constants; the potentials $a(r)$ and $b(r)$ are defined by the metric (90).

approximations than the first order⁴ with respect to the quantities GM/\bar{r} and \bar{v}^2 assumed small with respect to the squared light speed. This approximation is sometimes referred to as an expansion in inverse powers of the light speed.

The typical values of the Newtonian gravitational potential Φ are nowhere larger (in modulus) than 10^{-5} in the Solar System (in geometrized units, Φ is dimensionless). On the other hand, planetary velocities satisfy the condition $\bar{v}^2 \lesssim -\Phi$, while the matter pressure p , experienced inside the Sun and the planets, is generally smaller than the matter gravitational energy density $-\rho\Phi$, in other words⁵ $p/\rho \lesssim -\Phi$. Furthermore one has to consider that even other forms of energy in the Solar System (compressional energy, radiation, thermal energy, etc.) have small intensities and the specific energy density Π (the ratio of the energy density to the rest-mass density) is related to U by $\Pi \lesssim U$ (Π is $\sim 10^{-5}$ in the Sun and $\sim 10^{-9}$ in the Earth [21]). As matter of fact, one can consider that these quantities, as function of the velocity, give second order contributions:

$$-\Phi \sim v^2 \sim p/\rho \sim \Pi \sim \mathcal{O}(2) \quad (92)$$

Therefore, the velocity v gives $\mathcal{O}(1)$ terms in the velocity expansions, U^2 is of order $\mathcal{O}(4)$, Uv of $\mathcal{O}(3)$, $U\Pi$ is of $\mathcal{O}(4)$, and so on. Considering these approximations, one has

⁴ This approximation coincides with the Newtonian Mechanics.

⁵ Typical values of p/ρ are $\sim 10^{-5}$ in the Sun and $\sim 10^{-10}$ in the Earth [21].

$$\frac{\partial}{\partial t} \sim \mathbf{v} \cdot \nabla \quad (93)$$

and

$$\frac{|\partial/\partial t|}{|\nabla|} \sim \mathcal{O}(1) \quad (94)$$

The paradigm of post-Newtonian limit is to start from a series-developed metric tensor (and all additional fields in the theory) with respect to the dimensionless quantity v (in natural units!). A system of moving bodies radiates gravitational waves and thus loses energy. This loss appears only at the fifth-order in the approximation of v . In the first four approximations, the energy of the system remains constant. From this, it follows that a system of gravitating bodies can be correctly described by a Lagrangian up to terms of order v^4 , in the absence of electromagnetic fields. We thus find the equations of motion of the system in the next approximation after the Newtonian.

To solve the problem, we have to start from the determination of the metric tensor $g_{\mu\nu}$ in the same approximation of the gravitational field [45, 46].

Considering that particles move along the geodesics

$$\frac{d^2 x^\mu}{ds^2} + \Gamma_{\sigma\tau}^\mu \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} = 0 \quad (95)$$

these can be written in details as

$$\frac{d^2 x^i}{dt^2} = -\Gamma_{tt}^i - 2\Gamma_{tm}^i \frac{dx^m}{dt} - \Gamma_{mn}^i \frac{dx^m}{dt} \frac{dx^n}{dt} + \left[\Gamma_{tt}^t + 2\Gamma_{tm}^t \frac{dx^m}{dt} + 2\Gamma_{mn}^t \frac{dx^m}{dt} \frac{dx^n}{dt} \right] \frac{dx^i}{dt} \quad (96)$$

In the Newtonian approximation, that is vanishingly small velocities and only first-order terms in the difference between $g_{\mu\nu}$ and the Minkowski metric $\eta_{\mu\nu}$, one obtains that the particle motion equations reduce to the standard result

$$\frac{d^2 x^i}{dt^2} \simeq -\Gamma_{tt}^i \simeq -\frac{1}{2} \frac{\partial g_{tt}}{\partial x^i} \quad (97)$$

The quantity $1 - g_{tt}$ is of order $G\bar{M}/\bar{r}$, so that the Newtonian approximation gives $\frac{d^2 x^i}{dt^2}$ to the order $G\bar{M}/\bar{r}^2$, that is, to the order \bar{v}^2/\bar{r} . As a consequence, if we would like to search for the post-Newtonian approximation, we need to compute $\frac{d^2 x^i}{dt^2}$ to the order \bar{v}^4/\bar{r} . Due to the Equivalence Principle and the differentiability of spacetime manifold, we expect that it should be possible to find out a coordinate system in which the metric tensor is nearly equal to the Minkowski one $\eta_{\mu\nu}$, the correction being expandable in powers of $G\bar{M}/\bar{r} \sim \bar{v}^2$. In other words, one has to consider the metric developed as follows:

$$\begin{cases} g_{tt}(t, \mathbf{x}) \simeq 1 + g_{tt}^{(2)}(t, \mathbf{x}) + g_{tt}^{(4)}(t, \mathbf{x}) + \mathcal{O}(6) \\ g_{ti}(t, \mathbf{x}) \simeq g_{ti}^{(3)}(t, \mathbf{x}) + \mathcal{O}(5) \\ g_{ij}(t, \mathbf{x}) \simeq -\delta_{ij} + g_{ij}^{(2)}(t, \mathbf{x}) + \mathcal{O}(4) \end{cases} \quad (98)$$

where δ_{ij} is the Kronecker delta, and for the contravariant form of $g_{\mu\nu}$, one has

$$\begin{cases} g^{tt}(t, \mathbf{x}) \simeq 1 + g^{(2)tt}(t, \mathbf{x}) + g^{(4)tt}(t, \mathbf{x}) + \mathcal{O}(6) \\ g^{ti}(t, \mathbf{x}) \simeq g^{(3)ti}(t, \mathbf{x}) + \mathcal{O}(5) \\ g^{ij}(t, \mathbf{x}) \simeq -\delta_{ij} + g^{(2)ij}(t, \mathbf{x}) + \mathcal{O}(4) \end{cases} \quad (99)$$

The inverse of the metric tensor (98) is defined by (1). The relations among the higher than first order terms turn out to be

$$\begin{cases} g^{(2)tt}(t, \mathbf{x}) = -g_{tt}^{(2)}(t, \mathbf{x}) \\ g^{(4)tt}(t, \mathbf{x}) = g_{tt}^{(2)}(t, \mathbf{x})^2 - g_{tt}^{(4)}(t, \mathbf{x}) \\ g^{(3)ti} = g_{ti}^{(3)} \\ g^{(2)ij}(t, \mathbf{x}) = -g_{ij}^{(2)}(t, \mathbf{x}) \end{cases} \quad (100)$$

In evaluating $\Gamma_{\alpha\beta}^\mu$ we must take into account that the scale of distance and time, in our systems, are respectively set by \bar{r} and \bar{r}/\bar{v} , thus the space and time derivatives should be regarded as being of order

$$\frac{\partial}{\partial x^i} \sim \frac{1}{\bar{r}}, \quad \frac{\partial}{\partial t} \sim \frac{\bar{v}}{\bar{r}} \quad (101)$$

Using the above approximations (98), (99) and (100) we have, from the definition (3),

$$\begin{cases} \Gamma^{(3)t}_{tt} = \frac{1}{2}g_{tt,t}^{(2)} & \Gamma^{(2)i}_{tt} = \frac{1}{2}g_{tt,i}^{(2)} \\ \Gamma^{(2)i}_{jk} = \frac{1}{2}\left(g_{jk,i}^{(2)} - g_{ij,k}^{(2)} - g_{ik,j}^{(2)}\right) & \Gamma^{(3)t}_{ij} = \frac{1}{2}\left(g_{ti,j}^{(3)} + g_{jt,i}^{(3)} - g_{ij,t}^{(2)}\right) \\ \Gamma^{(3)i}_{tj} = \frac{1}{2}\left(g_{tj,i}^{(3)} - g_{it,j}^{(3)} - g_{ij,t}^{(2)}\right) & \Gamma^{(4)t}_{ti} = \frac{1}{2}\left(g_{tt,i}^{(4)} - g_{tt}^{(2)}g_{ti,i}^{(2)}\right) \\ \Gamma^{(4)i}_{tt} = \frac{1}{2}\left(g_{tt,i}^{(4)} + g_{im}^{(2)}g_{tt,m}^{(2)} - 2g_{it,t}^{(3)}\right) & \Gamma^{(2)t}_{ti} = \frac{1}{2}g_{tt,i}^{(2)} \end{cases} \quad (102)$$

The Ricci tensor components (4) are

$$\begin{cases} R_{tt}^{(2)} = \frac{1}{2}g_{tt,mm}^{(2)} \\ R_{tt}^{(4)} = \frac{1}{2}g_{tt,mm}^{(4)} + \frac{1}{2}g_{mn,m}^{(2)}g_{tt,n}^{(2)} + \frac{1}{2}g_{mn}^{(2)}g_{tt,mn}^{(2)} + \frac{1}{2}g_{mm,tt}^{(2)} - \frac{1}{4}g_{tt,m}^{(2)}g_{tt,m}^{(2)} - \frac{1}{4}g_{mm,n}^{(2)}g_{tt,n}^{(2)} - g_{tm,tm}^{(3)} \\ R_{ti}^{(3)} = \frac{1}{2}g_{ti,mm}^{(3)} - \frac{1}{2}g_{im,mt}^{(2)} - \frac{1}{2}g_{mt,mi}^{(3)} + \frac{1}{2}g_{mm,ti}^{(2)} \\ R_{ij}^{(2)} = \frac{1}{2}g_{ij,mm}^{(2)} - \frac{1}{2}g_{im,mj}^{(2)} - \frac{1}{2}g_{jm,mi}^{(2)} - \frac{1}{2}g_{tt,ij}^{(2)} + \frac{1}{2}g_{mm,ij}^{(2)} \end{cases} \quad (103)$$

and the Ricci scalar (5) is

$$\begin{cases} R^{(2)} = R_{tt}^{(2)} - R_{mm}^{(2)} = g_{tt,mm}^{(2)} - g_{nn,mm}^{(2)} + g_{mn,mn}^{(2)} \\ R^{(4)} = R_{tt}^{(4)} - g_{tt}^{(2)}R_{tt}^{(2)} - g_{mn}^{(2)}R_{mn}^{(2)} = \\ = \frac{1}{2}g_{tt,mm}^{(4)} + \frac{1}{2}g_{mn,m}^{(2)}g_{tt,n}^{(2)} + \frac{1}{2}g_{mn}^{(2)}g_{tt,mn}^{(2)} + \frac{1}{2}g_{mm,tt}^{(2)} - \frac{1}{4}g_{tt,m}^{(2)}g_{tt,m}^{(2)} + \\ - \frac{1}{4}g_{mm,n}^{(2)}g_{tt,n}^{(2)} - g_{tm,tm}^{(3)} - \frac{1}{2}g_{tt}^{(2)}g_{tt,mm}^{(2)} - \frac{1}{2}g_{mn}^{(2)}\left(g_{mn,ll}^{(2)} - g_{ml,ln}^{(2)} - g_{nl,lm}^{(2)} - g_{tt,mn}^{(2)} + g_{ll,mn}^{(2)}\right) \end{cases} \quad (104)$$

The Einstein tensor components (10) are

$$\left\{ \begin{aligned} G_{tt}^{(2)} &= R_{tt}^{(2)} - \frac{1}{2}R^{(2)} = \frac{1}{2}g_{mm,nn}^{(2)} + \frac{1}{2}g_{mn,mn}^{(2)} \\ G_{tt}^{(4)} &= R_{tt}^{(4)} - \frac{1}{2}R^{(4)} - \frac{1}{2}g_{tt}^{(2)}R^{(2)} = \dots \\ G_{ti}^{(3)} &= R_{ti}^{(3)} = \frac{1}{2}g_{ti,mm}^{(3)} - \frac{1}{2}g_{im,mt}^{(2)} - \frac{1}{2}g_{mt,mi}^{(3)} + \frac{1}{2}g_{mm,ti}^{(2)} \\ G_{ij}^{(2)} &= R_{ij}^{(2)} + \frac{\delta_{ij}}{2}R^{(2)} = \frac{1}{2}g_{ij,mm}^{(2)} - \frac{1}{2}g_{im,mj}^{(2)} - \frac{1}{2}g_{jm,mi}^{(2)} - \frac{1}{2}g_{tt,ij}^{(2)} + \frac{1}{2}g_{mm,ij}^{(2)} + \frac{\delta_{ij}}{2}\left[g_{tt,mm}^{(2)} - g_{nn,mm}^{(2)} + g_{mn,mn}^{(2)}\right] \end{aligned} \right. \quad (105)$$

By assuming the harmonic gauge ⁶

$$g^{\rho\sigma}\Gamma_{\rho\sigma}^{\mu} = 0 \quad (106)$$

it is possible to simplify the components of Ricci tensor (103). In fact for $\mu = 0$ one has

$$2g^{\sigma\tau}\Gamma_{\sigma\tau}^t \approx g_{tt,t}^{(2)} - 2g_{tm,m}^{(3)} + g_{mm,t}^{(2)} = 0 \quad (107)$$

and for $\mu = i$

$$2g^{\sigma\tau}\Gamma_{\sigma\tau}^i \approx g_{tt,i}^{(2)} + 2g_{mi,m}^{(2)} - g_{mm,i}^{(2)} = 0 \quad (108)$$

Differentiating (107) with respect to t , x^j and (108) and with respect to t , one obtains

$$g_{tt,tt}^{(2)} - 2g_{tm,mt}^{(3)} + g_{mm,tt}^{(2)} = 0 \quad (109)$$

$$g_{tt,tj}^{(2)} - 2g_{mt,jm}^{(3)} + g_{mm,tj}^{(2)} = 0 \quad (110)$$

$$g_{tt,ti}^{(2)} + 2g_{mi,tm}^{(2)} - g_{mm,ti}^{(2)} = 0 \quad (111)$$

On the other side, combining (110) and (111), we get

$$g_{mm,ti}^{(2)} - g_{mi,tm}^{(2)} - g_{mt,mi}^{(3)} = 0 \quad (112)$$

Finally, differentiating (108) with respect to x^j , one has:

$$g_{tt,ij}^{(2)} + 2g_{mi,jm}^{(2)} - g_{mm,ij}^{(2)} = 0 \quad (113)$$

and redefining indexes as $j \rightarrow i$, $i \rightarrow j$ since these are mute indexes, we get

$$g_{tt,ij}^{(2)} + 2g_{mj,im}^{(2)} - g_{mm,ij}^{(2)} = 0 \quad (114)$$

⁶ The gauge transformation is $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \zeta_{\mu,\nu} - \zeta_{\nu,\mu}$ when we perform a coordinate transformation as $x'^{\mu} = x^{\mu} + \zeta^{\mu}$ with $O(\zeta^2) \ll 1$. To obtain our gauge and the validity of the field equations for both perturbation $h_{\mu\nu}$ and $\tilde{h}_{\mu\nu}$, the vector ζ_{μ} has to satisfy the harmonic condition $\square\zeta^{\mu} = 0$.

Combining (113) and (114), we obtain

$$g_{tt,ij}^{(2)} + g_{mi,jm}^{(2)} + g_{mj,im}^{(2)} - g_{mm,ij}^{(2)} = 0 \quad (115)$$

Relations (109), (112), (115) guarantee to rewrite the Ricci tensor components (103) at higher orders as

$$\left\{ \begin{array}{l} R_{tt}^{(2)}|_{HG} = \frac{1}{2}\Delta g_{tt}^{(2)} \\ R_{tt}^{(4)}|_{HG} = \frac{1}{2}\Delta g_{tt}^{(4)} + \frac{1}{2}g_{mn}^{(2)}g_{tt,mn}^{(2)} - \frac{1}{2}g_{tt,tt}^{(2)} - \frac{1}{2}|\nabla g_{tt}^{(2)}|^2 \\ R_{ti}^{(3)}|_{HG} = \frac{1}{2}\Delta g_{ti}^{(3)} \\ R_{ij}^{(2)}|_{HG} = \frac{1}{2}\Delta g_{ij}^{(2)} \end{array} \right. \quad (116)$$

and the Ricci scalar (104) becomes

$$\left\{ \begin{array}{l} R^{(2)}|_{HG} = \frac{1}{2}\Delta g_{tt}^{(2)} - \frac{1}{2}\Delta g_{mm}^{(2)} \\ R^{(4)}|_{HG} = \frac{1}{2}\Delta g_{tt}^{(4)} + \frac{1}{2}g_{mn}^{(2)}g_{tt,mn}^{(2)} - \frac{1}{2}g_{tt,tt}^{(2)} - \frac{1}{2}|\nabla g_{tt}^{(2)}|^2 - \frac{1}{2}g_{tt}^{(2)}\Delta g_{tt}^{(2)} - \frac{1}{2}g_{mn}^{(2)}\Delta g_{mn}^{(2)} \end{array} \right. \quad (117)$$

where ∇ and Δ are, respectively, the gradient and the Laplacian in flat space. The Einstein tensor components (10) in the harmonic gauge are

$$\left\{ \begin{array}{l} G_{tt}^{(2)}|_{HG} = \frac{1}{4}\Delta g_{tt}^{(2)} + \frac{1}{4}\Delta g_{mm}^{(2)} \\ G_{tt}^{(4)}|_{HG} = \dots \\ G_{ti}^{(3)}|_{HG} = \frac{1}{2}\Delta g_{ti}^{(3)} \\ G_{ij}^{(2)}|_{HG} = \frac{1}{2}\Delta g_{ij}^{(2)} + \frac{\delta_{ij}}{4} \left[\Delta g_{tt}^{(2)} - \Delta g_{mm}^{(2)} \right] \end{array} \right. \quad (118)$$

On the matter side, *i.e.* right-hand side of the field equations (9), we start with the general definition of the energy-momentum tensor of a perfect fluid (7) with additional energy Π

$$T_{\mu\nu} = (\rho + \Pi\rho + p)u_\mu u_\nu - p g_{\mu\nu} \quad (119)$$

The explicit form of the energy-momentum tensor can be derived as follows

$$\left\{ \begin{array}{l} T_{tt} = \rho + \rho(v^2 - 2U + \Pi) + \rho \left[v^2 \left(\frac{p}{\rho} + v^2 + 2V + \Pi \right) + \sigma - 2\Pi U \right] \\ T_{ti} = -\rho v^i + \rho \left[-v^i \left(\frac{p}{\rho} + 2V + v^2 + \Pi \right) + h_{ti} \right] \\ T_{ij} = \rho v^i v^j + p\delta_{ij} + \rho \left[v^i v^j \left(\Pi + \frac{p}{\rho} + 4V + v^2 + 2U \right) - 2v^c \delta_{c(i} h_{0|j)} + 2\frac{p}{\rho} V \delta_{ij} \right] \end{array} \right. \quad (120)$$

As a first application of these results, let us now take into account the simplest case, that is GR.

A. The Newtonian and Post-Newtonian Limit of General Relativity

Einstein Eqs. (9) can be rewritten as

$$R_{\mu\nu} = \mathcal{X} \left[T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right] \quad (121)$$

From the interpretation of stress-energy tensor components as energy density, momentum density and momentum flux, we have T_{tt} , T_{ti} and T_{ij} at the various orders

$$\begin{cases} T_{tt} = T_{tt}^{(0)} + T_{tt}^{(2)} + \mathcal{O}(4) \\ T_{ti} = T_{ti}^{(1)} + \mathcal{O}(3) \\ T_{ij} = T_{ij}^{(2)} + \mathcal{O}(4) \end{cases} \quad (122)$$

where $T_{\mu\nu}^{(N)}$ denotes the term in $T_{\mu\nu}$ of order $\bar{M}/\bar{r}^3 \bar{v}^N$. In particular $T_{tt}^{(0)}$ is the density of rest-mass, while $T_{tt}^{(2)}$ is the non-relativistic part of the energy density. What we need is the tensor

$$S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \quad (123)$$

$G\bar{M}/\bar{r}$ is of order \bar{v}^2 , so (98) and (122) give

$$\begin{cases} S_{tt} = S_{tt}^{(0)} + S_{tt}^{(2)} + \mathcal{O}(6) \\ S_{ti} = S_{ti}^{(1)} + \mathcal{O}(3) \\ S_{ij} = S_{ij}^{(0)} + \mathcal{O}(2) \end{cases} \quad (124)$$

where $S_{\mu\nu}^{(N)}$ denotes the term in $S_{\mu\nu}$ of order $\bar{M}/\bar{r}^3 \bar{v}^N$. In particular

$$\begin{cases} S_{tt}^{(0)} = \frac{1}{2} T_{tt}^{(0)} \\ S_{tt}^{(2)} = \frac{1}{2} T_{tt}^{(2)} + \frac{1}{2} T_{mm}^{(2)} \\ S_{ti}^{(1)} = T_{ti}^{(1)} \\ S_{ij}^{(0)} = \frac{1}{2} \delta_{ij} T_{tt}^{(0)} \end{cases} \quad (125)$$

Substituting Eqs.(116) and (124) in Eqs.(121), we find that the field equations in harmonic coordinates are indeed consistent with the expansions we are using, and give

$$\begin{cases} R_{tt}^{(2)} = \mathcal{X} S_{tt}^{(0)} \\ R_{tt}^{(4)} = \mathcal{X} S_{tt}^{(2)} \\ R_{ti}^{(3)} = \mathcal{X} S_{ti}^{(0)} \\ R_{ij}^{(2)} = \mathcal{X} S_{ij}^{(0)} \end{cases} \quad (126)$$

and, in particular,

$$\begin{cases} \Delta g_{tt}^{(2)} = \mathcal{X} T_{tt}^{(0)} \\ \Delta g_{tt}^{(4)} = \mathcal{X} \left[T_{tt}^{(2)} + T_{mm}^{(2)} \right] - g_{mn}^{(2)} g_{tt,mn}^{(2)} + g_{tt,tt}^{(2)} + |\nabla g_{tt}^{(2)}|^2 \\ \Delta g_{ti}^{(3)} = 2 \mathcal{X} T_{ti}^{(1)} \\ \Delta g_{ij}^{(2)} = \mathcal{X} \delta_{ij} T_{tt}^{(0)} \end{cases} \quad (127)$$

From the first one of (127), we find, as expected, the Newtonian result:

$$g_{tt}^{(2)} = -\frac{\mathcal{X}}{4\pi} \int d^3 \mathbf{x}' \frac{T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = -2G \int d^3 \mathbf{x}' \frac{T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \doteq 2\Phi(\mathbf{x}) \quad (128)$$

where $\Phi(\mathbf{x})$ is the gravitational potential. In fact if we consider a point-like source (51) we find

$$\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x}|} \quad (129)$$

From the third and fourth equations of system (127), we find that

$$\begin{cases} g_{ti}^{(3)} = -\frac{\mathcal{X}}{2\pi} \int d^3 \mathbf{x}' \frac{T_{ti}^{(1)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \doteq Z_i(\mathbf{x}) \\ g_{ij}^{(2)} = -\frac{\mathcal{X}}{4\pi} \delta_{ij} \int d^3 \mathbf{x}' \frac{T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = 2\delta_{ij} \Phi(\mathbf{x}) \end{cases} \quad (130)$$

The second equation of (127) can be rewritten as follows

$$\Delta \left[g_{tt}^{(4)} - 2\Phi^2 \right] = \mathcal{X} \left[T_{tt}^{(2)} + T_{mm}^{(2)} \right] - 8\Phi \Delta \Phi + 2\Phi_{,tt} \quad (131)$$

and the solution for $g_{tt}^{(4)}$ is

$$g_{tt}^{(4)} = 2\Phi^2 - \frac{\mathcal{X}}{4\pi} \int d^3 \mathbf{x}' \frac{T_{tt}^{(2)}(\mathbf{x}') + T_{mm}^{(2)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{2}{\pi} \int d^3 \mathbf{x}' \frac{\Phi(\mathbf{x}') \Delta_{\mathbf{x}'} \Phi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{2\pi} \int d^3 \mathbf{x}' \frac{\Phi_{,tt}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \doteq 2\Upsilon(\mathbf{x}) \quad (132)$$

By using the equations at second order, we obtain the final expression for the correction at fourth order in the time-time component of the metric:

$$\Upsilon(\mathbf{x}) = \Phi(\mathbf{x})^2 - \frac{\mathcal{X}}{8\pi} \int d^3 \mathbf{x}' \frac{T_{tt}^{(2)}(\mathbf{x}') + T_{mm}^{(2)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{\mathcal{X}}{\pi} \int d^3 \mathbf{x}' \frac{\Phi(\mathbf{x}') T_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi} \partial_{tt}^2 \int d^3 \mathbf{x}' \frac{\Phi(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (133)$$

We can rewrite the metric expression (98) as follows

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + 2\Phi + 2\Upsilon & \vec{Z}^T \\ \vec{Z} & -\delta_{ij}(1 - 2\Phi) \end{pmatrix} \quad (134)$$

where \vec{Z} are higher order terms that can be assumed null at this approximation level.

Finally the Lagrangian of a particle in presence of a gravitational field can be expressed as proportional to the invariant distance $ds^{1/2}$, thus we have:

$$L = \left(g_{\rho\sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} \right)^{1/2} = \left(g_{tt} + 2g_{tm}v^m + g_{mn}v^m v^n \right)^{1/2} = \left(1 + g_{tt}^{(2)} + g_{tt}^{(4)} + 2g_{tm}^{(3)}v^m - \mathbf{v}^2 + g_{mn}^{(2)}v^m v^n \right)^{1/2} \quad (135)$$

which, to the $\mathcal{O}(2)$ order, reduces to the classic Newtonian Lagrangian of a test particle $L_{\text{New}} = \left(1 + 2\Phi - \mathbf{v}^2 \right)^{1/2}$, where $v^m = \frac{dx^m}{dt}$ and $|\mathbf{v}|^2 = v^m v_m$. As matter of fact, post-Newtonian physics has to involve higher than $\mathcal{O}(2)$ order terms in the Lagrangian. In fact we obtain

$$L \sim 1 + \left[\Phi - \frac{1}{2}\mathbf{v}^2 \right] + \frac{3}{4} \left[\Upsilon + Z_m v^m + \Phi \mathbf{v}^2 \right] \quad (136)$$

An important remark concerns the odd-order perturbation terms $\mathcal{O}(1)$ or $\mathcal{O}(3)$. Since, these terms contain odd powers of velocity \mathbf{v} or of time derivatives, they are related to the energy dissipation or absorption by the system. Nevertheless, the mass-energy conservation prevents the energy and mass losses and, as a consequence, prevents, in the Newtonian limit, terms of $\mathcal{O}(1)$ and $\mathcal{O}(3)$ orders in the Lagrangian. If one takes into account contributions higher than $\mathcal{O}(4)$ order, different theories give different predictions. GR, for example, due to the conservation of post-Newtonian energy, forbids terms of $\mathcal{O}(5)$ order; on the other hand, terms of $\mathcal{O}(7)$ order can appear and are related to the energy lost by means of the gravitational radiation.

V. THE NEWTONIAN LIMIT OF $f(R)$ -GRAVITY BY THE O'HANLON THEORY ANALOGY

Let us start our analysis of Newtonian and post-Newtonian limits of Extended Theories of Gravity discussing the possible shortcomings related to the use of analogies in the weak field limit approximation. As briefly pointed out above, some authors have claimed that FOG models are characterized by an ill defined behavior in the Newtonian regime. In particular, in a series of papers [47] it is addressed that post-Newtonian corrections of the gravitational potential violate experimental constraints since these quantities can be recovered by a direct analogy with Brans-Dicke Gravity [50] simply supposing the Brans-Dicke parameter ω_{BD} in Eq. (25) vanishing for $f(R)$ -gravity. Actually, despite the calculations of the Newtonian and the post-Newtonian limit of $f(R)$ -gravity, performed in a rigorous manner, have showed that this is not the case [44, 51–53], it remains to clarify why the analogy with Brans-Dicke Gravity seems to fail its predictions. The issue is easily overcome once the correct analogy between $f(R)$ -gravity and the Brans-Dicke theory is taken into account.

It can be easily shown that, $f(R)$ -gravity models can be rewritten in term of a scalar-field Lagrangian non minimally coupled with gravity but without any kinetic term implying $\omega_{BD} = 0$. Actually, the simplest case of Scalar-Tensor Gravity models has been introduced some decades ago by Brans and Dicke in order to give a general mechanism capable of explaining the inertial forces by means of a background gravitational interaction. The explicit expression of such gravitational action is (25), while the general action of $f(R)$ -gravity is (18) when $f(X, Y, Z, \dots) = f(R)$. As said above, $f(R)$ -gravity can be recast as a Scalar-Tensor theory by introducing a suitable scalar field ϕ which non-minimally couples with the gravity sector. It is important to remark that such an analogy holds in a formalism in which the scalar field displays no kinetic term but it is characterized by means of a self-interaction potential which determines the whole dynamics (*O'Hanlon Lagrangian*) [54]. We can resume the actions as follow

$$\left\{ \begin{array}{l} \mathcal{A}_{JF}^{f(R)} = \int d^4x \sqrt{-g} \left[f(R) + \mathcal{X} \mathcal{L}_m \right] \\ \mathcal{A}_{JF}^{BD} = \int d^4x \sqrt{-g} \left[\phi R - \omega_{BD} \frac{\phi_{;\alpha} \phi^{;\alpha}}{\phi} + \mathcal{X} \mathcal{L}_m \right] \\ \mathcal{A}_{JF}^{OH} = \int d^4x \sqrt{-g} \left[\phi R + V(\phi) + \mathcal{X} \mathcal{L}_m \right] \end{array} \right. \quad (137)$$

where JF means that we are considering all theories in the Jordan frame. This consideration, therefore, implies that the scalar field Lagrangian equivalent to the purely geometrical $f(R)$ -gravity turns out to be quite different with

respect to the ordinary Brans-Dicke action in (25). This point represents a crucial aspect of our analysis. In fact, as we will show, such a difference implies completely different results in the Newtonian limit of the two models and, consequently, shows that it is misleading to extend predictions from the PPN approximation of Brans-Dicke models to $f(R)$ -gravity. Considering natural units, the O'Hanlon Lagrangian [54] is the third of (137). The field equations are obtained by varying the action with respect to both $g_{\mu\nu}$ and ϕ which now represent the dynamical variables (the same field equations are given setting $\omega(\phi) = 0$ and $F(\phi) = \phi$ in Eqs. (26)). Thus, one obtains

$$\begin{cases} \phi G_{\mu\nu} - \frac{1}{2}V(\phi)g_{\mu\nu} - \phi_{;\mu\nu} + g_{\mu\nu}\Box\phi = \mathcal{X}T_{\mu\nu} \\ R + \frac{dV(\phi)}{d\phi} = 0 \\ \Box\phi + \frac{1}{3}\phi\frac{dV(\phi)}{d\phi} - \frac{2}{3}V(\phi) = \frac{\mathcal{X}}{3}T \end{cases} \quad (138)$$

where the second line of (138) is the field equation for ϕ . While the third equation is a combination of the trace of the first and the second ones. Field equations for $f(R)$ -gravity are obtained from (18)

$$\begin{cases} f'R_{\mu\nu} - \frac{f}{2}g_{\mu\nu} - f'_{;\mu\nu} + g_{\mu\nu}\Box f' = \mathcal{X}T_{\mu\nu} \\ 3\Box f' + f'R - 2f = \mathcal{X}T \end{cases} \quad (139)$$

The two approaches can be mapped one into the other considering the following equivalences

$$\begin{cases} \phi = f' \\ V(\phi) = f - f'R \\ \phi\frac{dV(\phi)}{d\phi} - 2V(\phi) = f'R - 2f \end{cases} \quad (140)$$

where the Jacobian matrix of the transformation $\phi \iff f'$ has to be non-vanishing. Henceforth we can consider instead of (139) a new set of field equations determined by the equivalence of $f(R)$ -gravity with the O'Hanlon approach [55]

$$\begin{cases} \phi R_{\mu\nu} + \frac{1}{6}\left(V(\phi) + \phi\frac{dV(\phi)}{d\phi}\right)g_{\mu\nu} - \phi_{;\mu\nu} = \mathcal{X}\Sigma_{\mu\nu} \\ \Box\phi + \frac{1}{3}\left(\phi\frac{dV(\phi)}{d\phi} - 2V(\phi)\right) = \frac{\mathcal{X}}{3}T \end{cases} \quad (141)$$

where $\Sigma_{\mu\nu} = T_{\mu\nu} - \frac{1}{3}Tg_{\mu\nu}$. Let us, now, calculate the Newtonian limit of field equations (141). We take into account the perturbations of metric tensor $g_{\mu\nu}$ in Eqs.(98) up to $\mathcal{O}(2)$ -order and also for scalar field ϕ an analogous perturbation with respect to the background value

$$\phi \sim \phi^{(0)} + \phi^{(2)} \quad (142)$$

The differential operators turn out to be approximated as

$$\Box \approx \partial_t^2 - \Delta \quad \text{and} \quad \nabla_\mu \nabla_\nu \approx \partial_{\mu\nu}^2 \quad (143)$$

Actually in order to simplify calculations we can exploit the intrinsic gauge freedom in the metric definition. In particular, we choose the harmonic gauge (106) and the expressions of Ricci tensor components are given by (116). In relation to the adopted approximation we coherently develop the self-interaction potential at second order. In particular, the quantities in (141) read

$$\begin{cases} \phi V(\phi) + \phi \frac{dV(\phi)}{d\phi} \simeq V(\phi^{(0)}) + \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} + \left[\phi^{(0)} \frac{d^2 V(\phi^{(0)})}{d\phi^2} + 2 \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)} \\ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \simeq \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} - 2V(\phi^{(0)}) + \left[\phi^{(0)} \frac{d^2 V(\phi^{(0)})}{d\phi^2} - \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)} \end{cases} \quad (144)$$

The field equations (141), solved at $\mathcal{O}(0)$ -order of approximation, provide the two solutions

$$V(\phi^{(0)}) = 0 \quad \text{and} \quad \frac{dV(\phi^{(0)})}{d\phi} = 0 \quad (145)$$

which fix the $\mathcal{O}(0)$ -order terms in the development of the self-interaction potential; therefore we have

$$\begin{cases} V(\phi) + \phi \frac{dV(\phi)}{d\phi} \simeq \phi^{(0)} \frac{d^2 V(\phi^{(0)})}{d\phi^2} \phi^{(2)} \doteq 3m^2 \phi^{(2)} \\ \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \simeq \phi^{(0)} \frac{\delta^2 V(\phi^{(0)})}{d\phi^2} \phi^{(2)} \doteq 3m^2 \phi^{(2)} \end{cases} \quad (146)$$

where constant factors $\phi^{(0)} \frac{d^2 V(\phi^{(0)})}{d\phi^2}$ have been condensed within the quantity $3m^2$ ⁷. Such a constant can be easily interpreted as a mass term as will become clearer in the following. Now, taking into account the above simplifications, we can rewrite field Eqs. (141) at the $\mathcal{O}(2)$ -order in the form

$$\Delta g_{tt}^{(2)} = \frac{2\mathcal{X}}{\phi^{(0)}} \Sigma_{tt}^{(0)} - m^2 \frac{\phi^{(2)}}{\phi^{(0)}} \quad (147)$$

$$\Delta g_{ij}^{(2)} = \frac{2\mathcal{X}}{\phi^{(0)}} \Sigma_{ij}^{(0)} + m^2 \frac{\phi^{(2)}}{\phi^{(0)}} \delta_{ij} + 2 \frac{\phi_{,ij}^{(2)}}{\phi^{(0)}} \quad (148)$$

$$\Delta \phi^{(2)} - m^2 \phi^{(2)} = -\frac{\mathcal{X}}{3} T^{(0)} \quad (149)$$

The scalar field solution can be easily obtained from Eq. (149) as

$$\phi(\mathbf{x}) = \phi^{(0)} + \frac{\mathcal{X}}{3} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{\tilde{T}^{(0)}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}}{\mathbf{k}^2 + m^2} \quad (150)$$

where $\tilde{T}^{(0)}(\mathbf{k})$ is the Fourier transform of the trace $T^{(0)}$. While for $g_{tt}^{(2)}$ and $g_{ij}^{(2)}$ we have

$$g_{tt}^{(2)}(\mathbf{x}) = -\frac{\mathcal{X}}{2\pi\phi^{(0)}} \int d^3 \mathbf{x}' \frac{\Sigma_{tt}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{m^2}{4\pi\phi^{(0)}} \int d^3 \mathbf{x}' \frac{\phi^{(2)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (151)$$

$$\begin{aligned} g_{ij}^{(2)}(\mathbf{x}) = & -\frac{\mathcal{X}}{2\pi\phi^{(0)}} \int d^3 \mathbf{x}' \frac{\Sigma_{ij}^{(0)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{m^2 \delta_{ij}}{4\pi\phi^{(0)}} \int d^3 \mathbf{x}' \frac{\phi^{(2)}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ & + \frac{2}{\phi^{(0)}} \left[\frac{x_i x_j}{\mathbf{x}^2} \phi^{(2)}(\mathbf{x}) + \left(\delta_{ij} - \frac{3x_i x_j}{\mathbf{x}^2} \right) \frac{1}{|\mathbf{x}|^3} \int_0^{|\mathbf{x}|} d|\mathbf{x}'| |\mathbf{x}'|^2 \phi^{(2)}(\mathbf{x}') \right] \end{aligned} \quad (152)$$

⁷ The factor 3 is introduced to simplify an analogous factor present in the field equations (141).

The above three solutions represent a completely general result. In particular adopting transformations (140), one can straightforwardly obtain the solutions in the $f(R)$ -scheme.

Let us analyze the above results with an example. We can consider a FOG Lagrangian of the form

$$f(R) = a_1 R + a_2 R^2 \quad (153)$$

so that the scalar field reads $\phi = a_1 + 2a_2 R$ (a_1 and a_2 are arbitrary constants). The self-interaction potential turns out to be $V(\phi) = -\frac{(\phi-a_1)^2}{4a_2}$ satisfying the conditions $V(a_1) = 0$ and $V'(a_1) = 0$. In relation with the definition of the scalar field, we can opportunely identify a_1 with a constant value $\phi^{(0)} = a_1$. Furthermore, the scalar field "mass" can be expressed in term of the Lagrangian parameters as follows

$$m^2 = \frac{1}{3}\phi^{(0)} \frac{\delta^2 V(\phi^{(0)})}{\delta \phi^2} = -\frac{a_1}{6a_2} \quad (154)$$

Since the Ricci scalar at the second order reads

$$R \simeq R^{(2)} = \frac{\phi^{(2)}}{2a_2} = \frac{\mathcal{X}}{6a_2} \int \frac{d^3 \mathbf{k}}{(2\pi)^{3/2}} \frac{\tilde{T}^{(0)}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}}}{\mathbf{k}^2 + m^2} \quad (155)$$

if we consider a point-like source (51) therefore we obtain

$$R^{(2)} = \frac{GM}{3\pi^2 a_2} \int d^3 \mathbf{k} \frac{e^{i\mathbf{k} \cdot \mathbf{x}}}{\mathbf{k}^2 + m^2} = -\sqrt{\frac{\pi}{2}} \frac{r_g m^2}{a_1} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \quad (156)$$

The immediate consequence is that the solution for the scalar field ϕ at second order is

$$\phi^{(2)} = 2a_2 R^{(2)} = \sqrt{\frac{\pi}{2}} \frac{r_g}{3} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \quad (157)$$

while the complete scalar field solution up to the second order of perturbation is given by

$$\phi = a_1 + \sqrt{\frac{\pi}{2}} \frac{r_g}{3} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \quad (158)$$

Once the behavior of the scalar field has been obtained up to the second order of perturbation, in the same way, one can deduce the expressions for $g_{tt}^{(2)}$ and $g_{ij}^{(2)}$, where $\Sigma_{tt}^{(0)} = \frac{2}{3}\rho$ and $\Sigma_{ij}^{(0)} = \frac{1}{3}\rho \delta_{ij} = \frac{1}{2}\Sigma_{tt}^{(0)} \delta_{ij}$. As matter of fact the metric solutions at the second order of perturbation are

$$\left\{ \begin{array}{l} g_{tt} = 1 - \frac{2}{3a_1} \frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}} \frac{1}{3a_1} \frac{r_g e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \\ g_{ij} = -\left\{ 1 + \frac{1}{3a_1} \frac{r_g}{|\mathbf{x}|} - \sqrt{\frac{\pi}{2}} \frac{r_g}{3a_1} \left[\left(\frac{1}{|\mathbf{x}|} - \frac{2}{m|\mathbf{x}|^2} - \frac{2}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} - \frac{2}{m^2|\mathbf{x}|^3} \right] \right\} \delta_{ij} \\ \quad + \frac{(2\pi)^{1/2} r_g}{3a_1} \left[\left(\frac{1}{|\mathbf{x}|} + \frac{3}{m|\mathbf{x}|^2} + \frac{3}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} - \frac{3}{m^2|\mathbf{x}|^3} \right] \frac{x_i x_j}{|\mathbf{x}|^2} \end{array} \right. \quad (159)$$

This quantity, which is directly related to the gravitational potential, shows that the gravitational potential of the O'Hanlon Lagrangian is non-Newtonian. Such a behavior prevents from obtaining a natural definition of the PPN parameters as corrections to the Newtonian potential. As matter of fact, since it is indeed not true that a generic $f(R)$ -gravity model corresponds to a Brans-Dicke model with $\omega_{BD} = 0$ coherently to its post-Newtonian approximation. In particular it turns out to be wrong considering the PPN parameter $\gamma = \frac{1+\omega_{BD}}{2+\omega_{BD}}$ (see, for example, [21]) of Brans - Dicke gravity and evaluating this at $\omega_{BD} = 0$ so that one gets $\gamma = 1/2$ as derived in [47].

Differently, because of the presence of the self-interaction potential $V(\phi)$, in the O'Hanlon Lagrangian, a Yukawa like correction appears in the Newtonian limit. As matter of fact, one obtains a different gravitational potential with respect to the standard Newtonian one and, as matter of fact, the fourth order corrections in term of the v/c ratio (Newtonian level), have to be evaluated in a complete new way. In other words, considering a Brans-Dicke Lagrangian and an O'Hanlon one, despite their similar structure, implies different predictions in the weak field and small velocity limits. Such a result represents a significant argument against the claim that FOG models can be ruled out only on the bases of the analogy with Brans-Dicke PPN parameters.

An important consideration is in order at this point on the meaning of PPN-parameters γ and β , defined as a correction to the Newtonian-like behavior of the gravitational potentials (69). Actually, if we consider the limit $f(R) \rightarrow R$, from (159) and set $a_1 = 2/3$ (a_1 is an arbitrary constant), we have

$$\begin{cases} g_{tt} = 1 - \frac{r_g}{|\mathbf{x}|} \\ g_{ij} = -\left(1 + \frac{1}{2} \frac{r_g}{|\mathbf{x}|}\right) \delta_{ij} \end{cases} \quad (160)$$

which suggest that the PPN parameter γ , in this limit, results $1/2$ which is in striking contrast with GR predictions ($\gamma \sim 1$). Such a result is not surprising. In fact, the GR limit of the O'Hanlon Lagrangian requires $\phi \sim \text{const}$ and $V(\phi) \rightarrow 0$ but such approximations induce mathematical inconsistencies in the field equations of $f(R)$ -gravity, once these have been obtained by a given O'Hanlon Lagrangian. Actually this is a general issue of O'Hanlon Lagrangian. In fact it can be demonstrated that field Eqs. (141) do not reduce to the standard GR ones (for $V(\phi) \rightarrow 0$ and $\phi \sim \text{const}$) since we have

$$\begin{cases} \phi R_{\mu\nu} + \frac{1}{6} \left(V(\phi) + \phi \frac{dV(\phi)}{d\phi} \right) g_{\mu\nu} - \phi_{;\mu\nu} = \mathcal{X} \Sigma_{\mu\nu} \\ \square \phi + \frac{1}{3} \left(\phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right) = \frac{\mathcal{X}}{3} T \end{cases} \rightarrow \begin{cases} R_{\mu\nu} = \frac{\mathcal{X}}{a_1} \Sigma_{\mu\nu} \\ 0 = \frac{\mathcal{X}}{3} T \end{cases} \quad (161)$$

In fact $\Sigma_{\mu\nu}$ components read $\Sigma_{tt} = \frac{2}{3}\rho$ and $\Sigma_{ij} = \frac{1}{3}\rho \delta_{ij} = \frac{1}{2}\Sigma_{tt} \delta_{ij}$ in place of $S_{tt} = \frac{1}{2}\rho$ and $S_{ij} = \frac{1}{2}\rho \delta_{ij} = S_{tt} \delta_{ij}$ as it should be for the GR field Eqs. (121). Such a pathology emerges also when the GR limit is performed from a pure Brans-Dicke Lagrangian. In such a case, in order to match the Hilbert-Einstein Lagrangian, one needs $\phi \sim \text{const}$ and $\omega_{BD} = 0$, the immediate consequence is that the PPN parameter γ turns out to be $1/2$, while it is well known that Brans-Dicke model fulfils low energy limit prescriptions in the limit $\omega \rightarrow \infty$. Even in this case, the problem, with respect to the GR predictions, is that the GR limit of the model introduces inconsistencies in the field equations. In other words, it is not possible to impose the same transformation which leads the Brans-Dicke theory into GR at the Lagrangian level on the solutions obtained by solving the field equations descending from the general Lagrangian. The relevant aspect of this discussion is that considering a $f(R)$ -model, in analogy with the O'Hanlon Lagrangian and supposing that the self-interaction potential is negligible, introduces a pathological behaviour in dynamics which results in obtaining a PPN parameter $\gamma = 1/2$. This is what happens when an effective approximation scheme is introduced in the field equations in order to calculate the weak field limit of FOG by means of Brans-Dicke model. Such a result seems, from another point of view, to enforce the claim that FOG models have to be carefully investigated in this limit and their analogy with scalar-tensor gravity should be opportunely considered.

A. Scalar-Tensor Gravity in Jordan and Einstein frames

Up to now, we have discussed the weak field and small velocity limit of FOG in term of Brans-Dicke like Lagrangian remaining in the Jordan frame. We have considered the weak field and small velocity limit when a conformal transformation (27) is applied to the O'Hanlon Lagrangian. Now we want to analyze the differences of considering a Scalar-Tensor theory in the Jordan frame and in the Einstein frame. The Scalar-Tensor action \mathcal{A}_{JF}^{ST} in the Jordan frame (24) is linked to the action \mathcal{A}_{EF}^{ST} in the Einstein frame (30) via the transformations (31) between the quantities in the two frames. The O'Hanlon theory in the Jordan frame is recovered from Eq.(24) imposing $F(\phi) = \phi$ and $\omega(\phi) = 0$. Action (30), in the Einstein frame results simplified and the transformation between the two scalar fields reads

$$\Omega(\varphi)d\varphi^2 = -\frac{3\Lambda}{2}\frac{d\phi^2}{\phi^2} \quad (162)$$

If we suppose $\Omega(\varphi) = -\Omega_0 < 0$ we have

$$\phi = k e^{\pm\lambda\varphi} \quad (163)$$

where $\lambda = \sqrt{\frac{2\Omega_0}{3\Lambda}}$ and k is an integration constant. The O'Hanlon theory, transformed in the Einstein frame, is

$$\mathcal{A}_{EF}^{OH} = \int d^4x \sqrt{-\tilde{g}} \left[\Lambda \tilde{R} - \Omega_0 \varphi_{;\alpha} \varphi^{;\alpha} + \frac{\Lambda^2}{k^2} e^{\mp 2\lambda\varphi} V(k e^{\pm\lambda\varphi}) + \frac{\mathcal{X}\Lambda^2}{k^2} e^{\mp 2\lambda\varphi} \mathcal{L}_m \left(\frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \right) \right] \quad (164)$$

The field equations are

$$\begin{cases} \Lambda \tilde{G}_{\mu\nu} - \frac{1}{2} \frac{\Lambda^2}{k^2} e^{\mp 2\lambda\varphi} V(k e^{\pm\lambda\varphi}) \tilde{g}_{\mu\nu} - \Omega_0 \left(\varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} \varphi_{;\alpha} \varphi^{;\alpha} \tilde{g}_{\mu\nu} \right) = \mathcal{X} \tilde{T}_{\mu\nu}^{\varphi} \\ 2\Omega_0 \tilde{\square} \varphi + \frac{\Lambda^2}{k^2} e^{\mp 2\lambda\varphi} \left[\frac{\delta V}{\delta \phi}(k e^{\pm\lambda\varphi}) \mp 2\lambda V(k e^{\pm\lambda\varphi}) \right] + \mathcal{X} \tilde{\mathcal{L}}_{m,\varphi} = 0 \\ \tilde{R} = -\frac{\mathcal{X}}{2\Lambda} \tilde{T}^{\varphi} + \frac{\Omega_0}{\Lambda} \varphi_{;\alpha} \varphi^{;\alpha} - \frac{2\Lambda}{k^2} e^{\mp 2\lambda\varphi} V(k e^{\pm\lambda\varphi}) \end{cases} \quad (165)$$

where the matter tensor, which now coupled with the scalar field φ , in the Einstein frame reads

$$\tilde{T}_{\mu\nu}^{\varphi} = \frac{-1}{\sqrt{-\tilde{g}}} \frac{\delta(\sqrt{-\tilde{g}} \tilde{\mathcal{L}}_m)}{\delta \tilde{g}^{\mu\nu}} = \frac{\Lambda^2}{2k^2} e^{\mp 2\lambda\varphi} \left[\mathcal{L}_m \left(\frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \right) \tilde{g}_{\mu\nu} - 2 \frac{\delta}{\delta \tilde{g}^{\mu\nu}} \mathcal{L}_m \left(\frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \right) \right] \quad (166)$$

and

$$\tilde{\mathcal{L}}_{m,\varphi} = \mp \frac{\Lambda^2 \lambda}{k^2} e^{\mp 2\lambda\varphi} \left[2\mathcal{L}_m \left(\frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \right) + \frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \frac{\delta \mathcal{L}_m}{\delta g_{\rho\sigma}} \left(\frac{\Lambda}{k} e^{\mp\lambda\varphi} \tilde{g}_{\rho\sigma} \right) \right] \quad (167)$$

Actually, in order to calculate the weak field and small velocity limit of the model in the Einstein frame, we can develop the two scalar fields at the second order $\phi \sim \phi^{(0)} + \phi^{(2)}$ and $\varphi \sim \varphi^{(0)} + \varphi^{(2)}$ with respect to a background value. This choice gives the relations

$$\begin{cases} \varphi^{(0)} = \pm \lambda^{-1} \ln \frac{\phi^{(0)}}{k} \\ \varphi^{(2)} = \pm \lambda^{-1} \frac{\phi^{(2)}}{\phi^{(0)}} \end{cases} \quad (168)$$

Let us consider the conformal transformation $\tilde{g}_{\mu\nu} = \frac{\phi}{\Lambda} g_{\mu\nu}$. From this relation, considering Eq.(163), one obtains for $\phi^{(0)} = \Lambda$

$$\begin{cases} \tilde{g}_{tt}^{(2)} = g_{tt}^{(2)} + \frac{\phi^{(2)}}{\phi^{(0)}} \\ \tilde{g}_{ij}^{(2)} = g_{ij}^{(2)} - \frac{\phi^{(2)}}{\phi^{(0)}} \delta_{ij} \end{cases} \quad (169)$$

As matter of fact, since $g_{tt}^{(2)} = 2\Phi^{JF}$, $g_{ij}^{(2)} = 2\Psi^{JF} \delta_{ij}$ and $\tilde{g}_{tt}^{(2)} = 2\Phi^{EF}$, $\tilde{g}_{ij}^{(2)} = 2\Psi^{EF} \delta_{ij}$ from Eqs.(169), relevant relations emerge linking the gravitational potentials between Jordan and Einstein frames

$$\begin{cases} \Phi^{EF} = \Phi^{JF} + \frac{\phi^{(2)}}{2\phi^{(0)}} = \Phi^{JF} \pm \frac{\lambda}{2}\varphi^{(2)} \\ \Psi^{EF} = \Psi^{JF} - \frac{\phi^{(2)}}{2\phi^{(0)}} = \Psi^{JF} \mp \frac{\lambda}{2}\varphi^{(2)} \end{cases} \quad (170)$$

If we introduce the variations of two potentials, $\Delta\Phi = \Phi^{JF} - \Phi^{EF}$ and $\Delta\Psi = \Psi^{JF} - \Psi^{EF}$, we obtain

$$\Delta\Phi = -\Delta\Psi = -\frac{\phi^{(2)}}{2\phi^{(0)}} = \mp \frac{\lambda}{2}\varphi^{(2)} \propto a_2 \propto f''(0) \quad (171)$$

here specified in the case of Lagrangian (153).

From the above expressions, one can notice that there is an evident difference between the behavior of the two gravitational potentials in the two frames [55]. Such a result suggests that, at the Newtonian level, it is possible to discriminate between the two frames thus one can deduce what is the true physical one. In particular, once, the gravitational potential is calculated in the Jordan frame and the dynamical evolution of ϕ is taken into account at the suitable perturbation level, this can be substituted in the first of (170) to obtain its Einstein frame evolution. The final step is that the two potentials have to be matched with experimental data in order to select what is the true physical solution.

VI. THE NEWTONIAN LIMIT OF $f(R)$ -GRAVITY IN STANDARD COORDINATES

The Newtonian limit of FOG can be worked out by comparing its viability with respect to the standard results of GR. Here, we investigate the limit in the metric approach, refraining from exploiting the formal equivalence of FOG with specific Scalar-Tensor theories, *i.e.* we work in the Jordan frame in order to avoid possible misleading interpretations of the results [45].

Considering the Taylor expansion of a generic $f(R)$ -gravity model, it is possible to obtain general solutions in term of the metric coefficients up to the third order of approximation. Furthermore, it is possible to show that the Birkhoff theorem is not a general result for $f(R)$ -gravity since time-dependent evolution of spherically symmetric solutions can be achieved depending on the order of perturbations.

Exploiting the formalism of Newtonian and post-Newtonian approximations previously described, we can develop a systematic analysis in the limits of weak field and small velocities for $f(R)$ -gravity. We are going to assume, as background, a spherically symmetric spacetime and we are going to investigate the vacuum case. Considering the metric (42), we have, for a given $g_{\mu\nu}$

$$\begin{cases} g_{tt}(t, r) \simeq 1 + g_{tt}^{(2)}(t, r) + g_{tt}^{(4)}(t, r) \\ g_{rr}(t, r) \simeq -1 + g_{rr}^{(2)}(t, r) \\ g_{\theta\theta}(t, r) = -r^2 \\ g_{\phi\phi}(t, r) = -r^2 \sin^2 \theta \end{cases} \quad (172)$$

while considering Eqs. (99), it is

$$\begin{cases} g^{tt} \simeq 1 - g_{tt}^{(2)} + [g_{tt}^{(2)^2} - g_{tt}^{(4)}] \\ g^{rr} \simeq -1 - g_{rr}^{(2)} \end{cases} \quad (173)$$

The determinant reads

$$g \simeq r^4 \sin^2 \theta \{-1 + [g_{rr}^{(2)} - g_{tt}^{(2)}] + [g_{tt}^{(2)} g_{rr}^{(2)} - g_{tt}^{(4)}]\} \quad (174)$$

Christoffel symbols (102) are

$$\left\{ \begin{array}{ll} \Gamma^{(3)t}_{tt} = \frac{g^{(2)}_{tt,t}}{2} & \Gamma^{(2)r}_{tt} + \Gamma^{(4)r}_{tt} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(2)}_{rr} g^{(2)}_{tt,r} + g^{(4)}_{tt,r}}{2} \\ \Gamma^{(3)r}_{tr} = -\frac{g^{(2)}_{rr,t}}{2} & \Gamma^{(2)t}_{tr} + \Gamma^{(4)t}_{tr} = \frac{g^{(2)}_{tt,r}}{2} + \frac{g^{(4)}_{tt,r} - g^{(2)}_{tt} g^{(2)}_{rr,r}}{2} \\ \Gamma^{(3)t}_{rr} = -\frac{g^{(2)}_{rr,t}}{2} & \Gamma^{(2)r}_{rr} + \Gamma^{(4)r}_{rr} = -\frac{g^{(2)}_{rr,r}}{2} - \frac{g^{(2)}_{rr} g^{(2)}_{rr,r}}{2} \\ \Gamma^r_{\phi\phi} = \sin^2 \theta \Gamma^r_{\theta\theta} & \Gamma^{(0)r}_{\theta\theta} + \Gamma^{(2)r}_{\theta\theta} + \Gamma^{(4)r}_{\theta\theta} = -r - r g^{(2)}_{rr} - r g^{(2)}_{rr}{}^2 \end{array} \right. \quad (175)$$

while the Ricci tensor components (103) are

$$\left\{ \begin{array}{l} R^{(2)}_{tt} = \frac{r g^{(2)}_{tt,rr} + 2g^{(2)}_{tt,r}}{2r} \\ R^{(4)}_{tt} = \frac{-r g^{(2)}_{tt,r}{}^2 + 4g^{(4)}_{tt,r} + r g^{(2)}_{tt,r} g^{(2)}_{rr,r} + 2g^{(2)}_{rr} [2g^{(2)}_{tt,r} + r g^{(2)}_{tt,rr}] + 2r g^{(4)}_{tt,rr} + 2r g^{(2)}_{rr,tt}}{4r} \\ R^{(3)}_{tr} = -\frac{g^{(2)}_{rr,t}}{r} \\ R^{(2)}_{rr} = -\frac{r g^{(2)}_{tt,rr} + 2g^{(2)}_{rr,r}}{2r} \\ R^{(2)}_{\theta\theta} = -\frac{2g^{(2)}_{rr} + r[g^{(2)}_{tt,r} + g^{(2)}_{rr,r}]}{2} \\ R^{(2)}_{\phi\phi} = \sin^2 \theta R^{(2)}_{\theta\theta} \end{array} \right. \quad (176)$$

and, finally, the Ricci scalar expression is

$$\left\{ \begin{array}{l} R^{(2)} = \frac{2g^{(2)}_{rr} + r[2g^{(2)}_{tt,r} + 2g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr}]}{r^2} \\ R^{(4)} = \frac{1}{2r^2} \left[4g^{(2)}_{rr}{}^2 + 2r g^{(2)}_{rr} [2g^{(2)}_{tt,r} + 4g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr}] + r \{ -r g^{(2)}_{tt,r}{}^2 + 4g^{(4)}_{tt,r} + \right. \\ \left. + r g^{(2)}_{tt,r} g^{(2)}_{rr,r} - 2g^{(2)}_{tt} [2g^{(2)}_{tt,r} + r g^{(2)}_{tt,rr}] + 2r g^{(4)}_{tt,rr} + 2r g^{(2)}_{rr,tt} \} \right] \end{array} \right. \quad (177)$$

By metric tensor (172) and by inserting it into the field equations (139), one obtains

$$\left\{ \begin{array}{l} H_{\mu\nu} = f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f'' \left\{ R_{,\mu\nu} - \Gamma^t_{\mu\nu} R_{,t} - \Gamma^r_{\mu\nu} R_{,r} - g_{\mu\nu} \left[\left(g^{tt}{}_{,t} + g^{tt} \ln \sqrt{-g}{}_{,t} \right) R_{,t} \right. \right. \\ \left. \left. + \left(g^{rr}{}_{,r} + g^{rr} \ln \sqrt{-g}{}_{,r} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] \right\} - f''' \left[R_{,\mu} R_{,\nu} - g_{\mu\nu} \left(g^{tt} R_{,t}{}^2 + g^{rr} R_{,r}{}^2 \right) \right] \\ H = f' R - 2f + 3f'' \left[\left(g^{tt}{}_{,t} + g^{tt} \ln \sqrt{-g}{}_{,t} \right) R_{,t} + \left(g^{rr}{}_{,r} + g^{rr} \ln \sqrt{-g}{}_{,r} \right) R_{,r} + g^{tt} R_{,tt} + g^{rr} R_{,rr} \right] \\ \left. + 3f''' \left[g^{tt} R_{,t}{}^2 + g^{rr} R_{,r}{}^2 \right] \right. \end{array} \right. \quad (178)$$

In order to derive the Newtonian and post-Newtonian approximations for a generic $f(R)$ -function, one should specify the $f(R)$ -Lagrangian into the field Eqs.(178). This is a crucial point because once a certain Lagrangian is chosen, one will obtain a particular approximation referred to such a choice. This means to lose any general prescription and to obtain corrections to the Newtonian potential, $\Phi(\mathbf{x})$, which refer "univocally" to the considered

$f(R)$ -function. Alternatively, one can restrict to analytic $f(R)$ -functions expandable with respect to a certain value $R = R_0 = \text{constant}$ or, at least, its non-analytic part, if exists at all, goes to zero faster than R^n , with $n \geq 2$ at $R \rightarrow 0$. In general, such theories are physically interesting and allow to recover the GR results and the correct boundary and asymptotic conditions. Then we assume

$$f(R) = \sum_n \frac{f^n(R_0)}{n!} (R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \dots \quad (179)$$

One has to note that the expansion (179), also if similar to (76), presents some differences. In fact $R^{(0)}$ is a general space-time function linked to the background metric $g_{\mu\nu}^{(0)}$ given in (74). Here R_0 is a constant value of scalar curvature, which can be negligible in weak field limit approximation. Besides, the coefficients f_0, f_1, f_2, f_3 are not proportional, respectively, to zero-th, first, second, third coefficient of the Taylor expansion of $f(R)$. In fact, we have

$$\begin{cases} f_0 = f(R_0) - R_0 f'(R_0) + \frac{1}{2} R_0^2 f''(R_0) - \frac{1}{6} R_0^3 f'''(R_0) \\ f_1 = f'(R_0) - R_0 f''(R_0) + \frac{1}{2} R_0^2 f'''(R_0) \\ f_2 = \frac{1}{2} f''(R_0) - \frac{1}{2} R_0 f'''(R_0) \\ f_3 = \frac{1}{6} f'''(R_0) \end{cases} \quad (180)$$

If we consider a flat background, then $R_0 = 0$ and the coefficients f_0, f_1, f_2, f_3 are the terms of Taylor series. But if we are searching for solutions at Newtonian and (possibility) post-Newtonian level, we have to consider a vanishing background scalar curvature. It is possible to obtain the Newtonian and post-Newtonian approximation of $f(R)$ -gravity considering such an expansion (179) into the field Eqs. (178) and to expand the system up to the orders $\mathcal{O}(0)$, $\mathcal{O}(2)$, $\mathcal{O}(3)$ and $\mathcal{O}(4)$. This approach provides general results and specific (analytic) Lagrangians are selected by the coefficients f_i in Eq.(179). Developing the equations in the case of vanishing matter, *i.e.* $T_{\mu\nu} = 0$, we have

$$\begin{cases} H_{\mu\nu}^{(0)} = 0, & H^{(0)} = 0 \\ H_{\mu\nu}^{(2)} = 0, & H^{(2)} = 0 \\ H_{\mu\nu}^{(3)} = 0, & H^{(3)} = 0 \\ H_{\mu\nu}^{(4)} = 0, & H^{(4)} = 0 \end{cases} \quad (181)$$

and, in particular, from the $\mathcal{O}(0)$ order approximation, one obtains

$$f_0 = 0 \quad (182)$$

which trivially follows from the above assumption that the space-time is asymptotically Minkowski (asymptotically flat background). This result suggests a first consideration. *If the Lagrangian is expanded around a vanishing value of the Ricci scalar ($R_0 = 0$), the relation (182) implies that the cosmological constant contribution has to be zero whatever is the $f(R)$ -gravity model.*

If we now consider the $\mathcal{O}(2)$ -order approximation, system (181), in the vacuum case, results to be

$$\begin{cases} f_1 r R^{(2)} - 2f_1 g_{tt,r}^{(2)} + 8f_2 R_{,r}^{(2)} - f_1 r g_{tt,rr}^{(2)} + 4f_2 r R^{(2)} = 0 \\ f_1 r R^{(2)} - 2f_1 g_{rr,r}^{(2)} + 8f_2 R_{,r}^{(2)} - f_1 r g_{tt,rr}^{(2)} = 0 \\ 2f_1 g_{rr}^{(2)} - r[f_1 r R^{(2)} - f_1 g_{tt,r}^{(2)} - f_1 g_{rr,r}^{(2)} + 4f_2 R_{,r}^{(2)} + 4f_2 r R_{,rr}^{(2)}] = 0 \\ f_1 r R^{(2)} + 6f_2 [2R_{,r}^{(2)} + r R_{,rr}^{(2)}] = 0 \\ 2g_{rr}^{(2)} + r[2g_{tt,r}^{(2)} - r R^{(2)} + 2g_{rr,r}^{(2)} + r g_{tt,rr}^{(2)}] = 0 \end{cases} \quad (183)$$

The last equation of the system (183) is the definition of Ricci scalar (5) at $\mathcal{O}(2)$ -order. The trace equation (the fourth line in Eqs. (183)), in particular, provides a differential equation with respect to the Ricci scalar which allows to solve, if $\text{sign}[f_1] = -\text{sign}[f_2]$, the system (183) at $\mathcal{O}(2)$ -order. The solutions are

$$\begin{cases} g_{tt}^{(2)} = \delta_0 - \frac{\delta_1}{f_1 r} + \frac{\delta_2(t)}{3m} \frac{e^{-mr}}{mr} + \frac{\delta_3(t)}{6m^2} \frac{e^{mr}}{mr} \\ g_{rr}^{(2)} = -\frac{\delta_1}{f_1 r} - \frac{\delta_2(t)}{3m} \frac{mr+1}{mr} e^{-mr} + \frac{\delta_3(t)}{6m^2} \frac{mr-1}{mr} e^{mr} \\ R^{(2)} = \delta_2(t) \frac{e^{-mr}}{r} + \frac{\delta_3(t)}{2m} \frac{e^{mr}}{r} \end{cases} \quad (184)$$

where

$$m \doteq \sqrt{-\frac{f_1}{6f_2}} \quad (185)$$

with the dimension of $length^{-1}$. We note that the definition of mass (185) is compatible with the definition of (154). Let us notice that the integration constant δ_0 has to be dimensionless, δ_1 has the dimension of $length$, while the time - dependent functions δ_2 and δ_3 , respectively, have the dimensions of $length^{-1}$ and $length^{-2}$. The functions $\delta_i(t)$ ($i = 2, 3$) are completely arbitrary since the differential equation system (183) contains only spatial derivatives. Besides, the integration constant δ_0 can be set to zero, as in the theory of the potential, since it represents an unessential additive quantity. When we consider the limit $f(R) \rightarrow R$, in the case of a point-like source (51), we recover the perturbed version of standard Schwarzschild solution (52) at $\mathcal{O}(2)$ -order with $\delta_1 = r_g$. In order to match at infinity the Minkowskian prescription of the metric, we discard the Yukawa growing mode present in (184). Then we have, in standard coordinates,

$$\begin{cases} ds^2 = \left[1 - \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m} \frac{e^{-mr}}{mr} \right] dt^2 - \left[1 + \frac{r_g}{f_1 r} + \frac{\delta_2(t)}{3m} \frac{mr+1}{mr} e^{-mr} \right] dr^2 - r^2 d\Omega \\ R = \frac{\delta_2(t) e^{-mr}}{r} \end{cases} \quad (186)$$

At this point one can provide the solution in term of the gravitational potential. In such a case, we have an explicit Newtonian-like term into the definition, according to previous results obtained with less rigorous methods [12, 56]. The first of (184) provides the second order solution in term of the metric expansion (see the definition (172)), but, this term coincides with the gravitational potential at the Newtonian order (128). In particular the gravitational potential of a FOG-model, analytic in the Ricci scalar R , is

$$\Phi = -\frac{GM}{f_1 r} + \frac{\delta_2(t)}{6m} \frac{e^{-mr}}{mr} \quad (187)$$

As first remark, one has to notice that the structure of the potential (187), for a given $f(R)$ -gravity, is determined only by the parameter m , (185), which depends on the first and the second derivative of the $f(R)$ -function, once developed around a vanishing value of the Ricci scalar.

As second remark, the solution for the gravitational potential has a Yukawa-like behavior depending on a characteristic length on which it evolves.

In other words, the correction to the Newtonian gravitational potential is always characterized by a Yukawa-like correction and only the first two terms of the Taylor expansion of a generic $f(R)$ -function turn out to be relevant. This is indeed a general result.

Let us now consider system (181) at third order approximation. The first important issue is that, at this order, one has to consider even the off-diagonal equation

$$f_1 g_{rr,t}^{(2)} + 2f_2 r R_{,tr}^{(2)} = 0 \quad (188)$$

which relates the time derivative of the Ricci scalar to the time derivative of $g_{rr}^{(2)}$. From this relation, it is possible to draw a relevant conclusion. One can deduce that, if the Ricci scalar depends on time, so it is the same for the metric components and even the gravitational potential turns out to be time-dependent. This result agrees with the analysis provided in [36] where a complete description of the weak field limit of FOG has been provided in term of the dynamical evolution of the Ricci scalar. Moreover it has been demonstrated that supposing the time independent Ricci scalar, a static spherically symmetric solution is found.

Eq. (188) confirms this result and provides the formal theoretical explanation of such a behavior. In particular, together with Eqs.(186), it suggests that if one considers the problem at a lower level of approximation (*i.e.* the second order) the background spacetime metric can be only factorized with a function space-depending and an arbitrary function time-depending. Then the Birkhoff theorem at Newtonian level is modified. The static solutions according to the Birkhoff theorem in GR are not directly obtained. Obviously this is still no more verified when the problem is faced with approximations of higher order. In other words, the debated issue to prove the validity of the Birkhoff theorem in FOG, finds here its physical answer. In [36], the validity of this theorem is demonstrated for FOG only when the Ricci scalar is time-independent or, in addition, when the solutions of (139) are investigated up to the second order of approximation in the metric coefficients (172). *Therefore, the Birkhoff theorem is not a general result for FOG but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time independent according to (186) and (187).*

The above results fix a fundamental difference between GR and FOG. While, in GR, a spherically symmetric solution represents a stationary and static configuration difficult to be related to a cosmological background evolution, this is no more true in the case of generic FOGs. In the latter case, a spherically symmetric background can show time-dependent evolution together with the radial dependence. In this sense, a relation between a spherical solution and the cosmological Hubble flow could be, in principle, achieved.

VII. THE NEWTONIAN AND POST-NEWTONIAN LIMIT OF $f(R)$ -GRAVITY IN ISOTROPIC COORDINATES

Let us consider now a form of metric tensor generalizing the metric (172). It is interesting, using the isotropic coordinates (41) and a more general approach, to solve the field equations as shown in [46]. The metric which we take into account is the following

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g_{tt}^{(2)}(t, \mathbf{x}) + g_{tt}^{(4)}(t, \mathbf{x}) + \dots & g_{ti}^{(3)}(t, \mathbf{x}) + \dots \\ g_{ti}^{(3)}(t, \mathbf{x}) + \dots & -\delta_{ij} + g_{ij}^{(2)}(t, \mathbf{x}) + \dots \end{pmatrix} \quad (189)$$

and the set of coordinates adopted is $x^\mu = (t, \mathbf{x}) = (t, x^1, x^2, x^3)$. The n -th derivative of function f can be developed as in Eq. (179) with $R_0 = 0$

$$f^n(R) \sim f^n(R^{(2)} + R^{(4)} + \dots) \sim f^n(0) + f^{n+1}(0)R^{(2)} + f^{n+1}(0)R^{(4)} + \frac{1}{2}f^{n+2}(0)R^{(2)^2} + \dots \quad (190)$$

From lowest order of field Eqs. (72), we find the same condition (182) ($f(0) = 0$). At $\mathcal{O}(2)$ -order (Newtonian level) Eqs. (72) becomes

$$\begin{cases} H_{tt}^{(2)} = f'(0)R_{tt}^{(2)} - \frac{f'(0)}{2}R^{(2)} - f''(0)\Delta R^{(2)} = \mathcal{X}T_{tt}^{(0)} \\ H_{ij}^{(2)} = f'(0)R_{ij}^{(2)} + \left[\frac{f'(0)}{2}R^{(2)} + f''(0)\Delta R^{(2)} \right] \delta_{ij} - f''(0)R_{,ij}^{(2)} = 0 \\ H^{(2)} = -3f''(0)\Delta R^{(2)} - f'(0)R^{(2)} = \mathcal{X}T^{(0)} \end{cases} \quad (191)$$

while at $\mathcal{O}(3)$ -order becomes

$$H_{ti}^{(3)} = f'(0)R_{ti}^{(3)} - f''(0)R^{(2)}_{,ti} = \mathcal{X}T_{ti}^{(1)} \quad (192)$$

Remembering the expressions of Christoffel symbols and using the following approximation for the determinant of metric tensor $\ln \sqrt{-g} \sim \frac{1}{2}[g_{tt}^{(2)} - g_{mm}^{(2)}] + \dots$, at $\mathcal{O}(4)$ -order, we have

$$\begin{cases} H_{tt}^{(4)} = f'(0)R_{tt}^{(4)} + f''(0)R^{(2)}R_{tt}^{(2)} - \frac{f'(0)}{2}R^{(4)} - \frac{f'(0)}{2}g_{tt}^{(2)}R^{(2)} - \frac{f''(0)}{4}R^{(2)^2} \\ \quad - f''(0)\left[g_{mn,m}^{(2)}R^{(2)}_{,n} + \Delta R^{(4)} + g_{tt}^{(2)}\Delta R^{(2)} + g_{mn}^{(2)}R^{(2)}_{,mn} - \frac{1}{2}\nabla g_{mm}^{(2)} \cdot \nabla R^{(2)} \right] \\ \quad - f'''(0)\left[|\nabla R^{(2)}|^2 + R^{(2)}\Delta R^{(2)} \right] = \mathcal{X}T_{tt}^{(2)} \\ H^{(4)} = -3f''(0)\Delta R^{(4)} - f'(0)R^{(4)} - 3f'''(0)\left[|\nabla R^{(2)}|^2 + R^{(2)}\Delta R^{(2)} \right] \\ \quad + 3f''(0)\left[R_{,tt}^{(2)} - g_{mn}^{(2)}R_{,mn}^{(2)} - \frac{1}{2}\nabla(g_{tt}^{(2)} - g_{mm}^{(2)}) \cdot \nabla R^{(2)} - g_{mn,m}^{(2)}R_{,n}^{(2)} \right] = \mathcal{X}T^{(2)} \end{cases} \quad (193)$$

Note that the propagation of Ricci scalar $R^{(4)}$ has the same dynamics of previous one (third line of Eqs. (191)). The complete knowledge of correction at fourth order for the tt -component of Ricci tensor fix the third derivative of $f(R)$ in $R = 0$. Also at this level, there is a degeneracy of $f(R)$ -theory: different theories, considering only the first three derivatives, admit the same gravitational field without radiation emission.

We want to rewrite and generalize the outcome of Eqs. (186) by introducing the Green function method (we remember that the Newtonian limit corresponds also to the linearization of field equations). Let us start from the trace equation. The solution for the Ricci scalar $R^{(2)}$ in the third line of Eqs.(191) is

$$R^{(2)}(t, \mathbf{x}) = \frac{m^2 \mathcal{X}}{f'(0)} \int d^3 \mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') T^{(0)}(t, \mathbf{x}') \quad (194)$$

where we defined

$$m^2 \doteq -\frac{f'(0)}{3f''(0)} \quad (195)$$

and $\mathcal{G}(\mathbf{x}, \mathbf{x}')$ is the Green function of field operator $\Delta - m^2$. We note that the definition of mass (195) is compatible with the definitions of (154) and (185).

The solution for $g_{tt}^{(2)}$, from the first line of (191) by considering that $R_{tt}^{(2)} = \frac{1}{2}\Delta g_{tt}^{(2)}$ (first line of (103) or (116)), is

$$g_{tt}^{(2)}(t, \mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3 \mathbf{x}' \frac{T_{tt}^{(0)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{4\pi} \int d^3 \mathbf{x}' \frac{R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} - \frac{2}{3m^2} R^{(2)}(t, \mathbf{x}) \quad (196)$$

We can check immediately that when $f(R) \rightarrow R$ we find $g_{tt}^{(2)}(t, \mathbf{x}) \rightarrow -2G \int d^3 \mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$. The expression (196) is the "modified" gravitational potential (here we have a factor 2) for $f(R)$ -gravity and generalize, for any matter distribution, the outcome (187). *This solution, which is the Newtonian limit of $f(R)$ -gravity, is also gauge-free.*

Since we have a linearized version of field equations, this limit corresponds to one of the Einstein equation and the linear superposition is satisfied. So the tt -component of energy-momentum tensor is, in this limit, the sum of mass-energy-volume density of sources, that is: $T_{tt}^{(0)} = \Sigma_a M_a \delta(\mathbf{x} - \mathbf{x}_a)$ where $\delta(\mathbf{x})$ is the delta function.

As it is evident, the Gauss theorem is not valid since the force law is not $\propto |\mathbf{x}|^{-2}$. The equivalence between a spherically symmetric distribution and a point-like distribution is not valid and how the matter is distributed in the space becomes extremely important in this situation. However, we have to say that the Bianchi identities hold in any case so the consistency of the theory is guaranteed.

From field Eq. (192), by using the gauge harmonic condition (106), we find the general solution for $g_{ti}^{(3)}$

$$g_{ti}^{(3)}(t, \mathbf{x}) = -\frac{\mathcal{X}}{2\pi f'(0)} \int d^3 \mathbf{x}' \frac{T_{ti}^{(1)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{1}{6\pi m^2} \frac{\partial}{\partial t} \int d^3 \mathbf{x}' \frac{\nabla_{i'} R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (197)$$

The choice of harmonic gauge enable us to solve Eq. (192) but we lose information on time evolution of $g_{tt}^{(2)}(t, \mathbf{x})$. This is important to obtain, at least in perturbative approach, some information about the Birkhoff theorem. By hypothesizing a perturbative approach (Newtonian-like), we confine time-evolution only to the time-variation of matter source. In fact in this working hypothesis, the motion of bodies embedded in gravitational fields evolves very slowly with respect to the internal motions of matter. Then we have, in any case, an instantaneous readjustment of spacetime. In other words, the motion of bodies is adiabatic and it enables us to factorize the solutions that, by a time transformation, become static solutions.

Still more, also the corrections to the gravito-magnetic effects (192) are depending on the only first two derivatives of $f(R)$ in $R = 0$. This means that different theories, from the third derivative on, admit the same Newtonian solution.

From second line of (191), by using the gauge harmonic condition (106), the solution for $g_{ij}^{(2)}$ follows

$$\begin{aligned} g_{ij}^{(2)}(t, \mathbf{x}) = & \left[\frac{1}{4\pi} \int d^3 \mathbf{x}' \frac{R^{(2)}(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} + \frac{2}{3m^2} R^{(2)}(t, \mathbf{x}) - \frac{1}{6\pi m^2} \frac{1}{|\mathbf{x}|^3} \int_{\Omega_{|\mathbf{x}|}} d^3 \mathbf{x}' R^{(2)}(t, \mathbf{x}') \right] \delta_{ij} \\ & + \left[\frac{1}{2\pi m^2 |\mathbf{x}|^3} \int_{\Omega_{|\mathbf{x}|}} d^3 \mathbf{x}' R^{(2)}(t, \mathbf{x}') - \frac{2}{3m^2} R^{(2)}(t, \mathbf{x}) \right] \frac{x_i x_j}{|\mathbf{x}|^2} \end{aligned} \quad (198)$$

where $\Omega_{|\mathbf{x}|}$ represents the integration volume with radius $|\mathbf{x}|$ (for the details see [26]). By the solutions (196), (197), (198) we can affirm that it is possible to have solution non-Ricci-flat in vacuum. This means that: *Higher Order Gravity mimics a matter source*. It is evident, from (196), that the Ricci scalar can be considered a "matter source" which curves the spacetime also in absence of ordinary matter. Then it is clear also that the knowledge of behavior of Ricci scalar inside mass distribution is fundamental to obtain the behavior of metric tensor outside the matter.

From the fourth order of field equations, we note also that the Ricci scalar $R^{(4)}$ propagates with the same m (the second line of (193)) and the solution at second order originates a supplementary matter source in *r.h.s.* of (72). The solution is

$$\begin{aligned} R^{(4)}(t, \mathbf{x}) = & \int d^3 \mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') \left\{ \frac{m^2 \mathcal{X}}{f'(0)} T^{(2)}(t, \mathbf{x}') - g_{mn,m}^{(2)}(t, \mathbf{x}') R_{,n}^{(2)}(t, \mathbf{x}') - g_{mn}^{(2)}(t, \mathbf{x}') R_{,mn}^{(2)}(t, \mathbf{x}') \right. \\ & + R_{,tt}^{(2)}(t, \mathbf{x}') - \frac{m^2}{\mu^4} \left[|\nabla_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}')|^2 + R^{(2)}(t, \mathbf{x}') \Delta_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}') \right] \\ & \left. - \frac{1}{2} \nabla_{\mathbf{x}'} \left[g_{tt}^{(2)}(t, \mathbf{x}') - g_{mm}^{(2)}(t, \mathbf{x}') \right] \cdot \nabla_{\mathbf{x}'} R^{(2)}(t, \mathbf{x}') \right\} \end{aligned} \quad (199)$$

where we introduced a new parameter

$$\mu^4 \doteq -\frac{f'(0)}{3f'''(0)} \quad (200)$$

Also in this case we can have a non-vanishing curvature in absence of matter. The solution for $g_{tt}^{(4)}$, from the first line of (193), is

$$\begin{aligned}
g_{tt}^{(4)}(t, \mathbf{x}) = & \int d^3\mathbf{x}' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \left\{ -\frac{\mathcal{X}T_{tt}^{(2)}(t, \mathbf{x}')}{2\pi f'(0)} + \frac{1}{6\pi\mu^4} \left[|\nabla R^{(2)}(t, \mathbf{x}')|^2 + R^{(2)}(t, \mathbf{x}') \Delta R^{(2)}(t, \mathbf{x}') \right] \right. \\
& + \frac{1}{4\pi} \left[g_{mn}^{(2)}(t, \mathbf{x}') g_{tt,mn}^{(2)}(t, \mathbf{x}') - g_{tt,tt}^{(2)}(t, \mathbf{x}') - |\nabla_{\mathbf{x}'} g_{tt}^{(2)}(t, \mathbf{x}')|^2 - R^{(4)}(t, \mathbf{x}') - g_{tt}^{(2)}(t, \mathbf{x}') R^{(2)}(t, \mathbf{x}') \right] \\
& + \frac{1}{6\pi m^2} \left[\frac{R^{(2)^2}(t, \mathbf{x}')}{4} - \frac{R^{(2)}(t, \mathbf{x}') \Delta g_{tt}^{(2)}(t, \mathbf{x}')}{2} + g_{mn,m}^{(2)}(t, \mathbf{x}') R^{(2)},_n(t, \mathbf{x}') + \Delta R^{(4)}(t, \mathbf{x}') \right. \\
& \left. \left. + g_{tt}^{(2)}(t, \mathbf{x}') \Delta R^{(2)}(t, \mathbf{x}') + g_{mn}^{(2)}(t, \mathbf{x}') R^{(2)},_{mn}(t, \mathbf{x}') - \frac{1}{2} \nabla g_{mm}^{(2)}(t, \mathbf{x}') \cdot \nabla R^{(2)}(t, \mathbf{x}') \right] \right\} \quad (201)
\end{aligned}$$

In summary, we have shown the more general solutions of field equations of $f(R)$ -gravity in the Newtonian and post-Newtonian limits assuming a coordinates transformation where the gauge harmonic condition is verified. Now we shall apply such an approach to obtain the explicit form of the metric tensor for a static and spherically symmetric matter source.

A. Solutions generated by an extended spherically symmetric source with harmonic gauge conditions

Let us consider a spherical source with mass M and radius ξ . Since the metric is given by Eqs.(189), the energy-momentum tensor (51) becomes

$$\begin{cases} T_{tt}(t, \mathbf{x}) \sim \rho(\mathbf{x}) + \rho(\mathbf{x}) g_{tt}^{(2)}(t, \mathbf{x}) = T_{tt}^{(0)}(t, \mathbf{x}) + T_{tt}^{(2)}(t, \mathbf{x}) \\ T = \rho(\mathbf{x}) = T^{(0)}(t, \mathbf{x}) \end{cases} \quad (202)$$

where the density is

$$\rho(\mathbf{x}) = \frac{3M}{4\pi\xi^3} \Theta(\xi - |\mathbf{x}|) \quad (203)$$

$\Theta(x)$ is the Heaviside function. We are not interested here to the internal structure of the source. The possible choices of the Green functions of the field operator $\Delta - m^2$, for spherically symmetric systems (*i.e.* $\mathcal{G}(\mathbf{x}, \mathbf{x}') = \mathcal{G}(|\mathbf{x} - \mathbf{x}'|)$), are the following

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = \begin{cases} -\frac{1}{4\pi} \frac{e^{-m|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{if } m^2 > 0 \\ C_1 \frac{e^{-im|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} + C_2 \frac{e^{im|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} & \text{if } m^2 < 0 \end{cases} \quad (204)$$

with $C_1 + C_2 = -\frac{1}{4\pi}$.

In the Newtonian limit of GR, the equation for the gravitational potential, generated by a point-like source

$$\Delta_{\mathbf{x}} \mathcal{G}_{New.mech.}(\mathbf{x} - \mathbf{x}') = -4\pi\delta(\mathbf{x} - \mathbf{x}') \quad (205)$$

is not satisfied by the Green functions (204). If we consider the flux of gravitational field $\mathbf{g}_{New.mech.}$ defined as

$$\mathbf{g}_{New.mech.} = -\frac{GM(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -GM \nabla_{\mathbf{x}} \mathcal{G}_{New.mech.}(\mathbf{x} - \mathbf{x}') \quad (206)$$

we obtain, as standard, the Gauss theorem

$$\int_{\Sigma} d\Sigma \quad \mathbf{g}_{New.mech.} \cdot \hat{n} \propto M \quad (207)$$

where Σ is a generic two-dimensional surface and \hat{n} its surface normal vector. The flux of field $\mathbf{g}_{New.mech.}$ on the surface Σ is proportional to the matter content M , inside the surface independently of the particular shape of surface (Gauss theorem, or Newton theorem for the gravitational field [57]). On the other hand, if we consider the flux defined by the new Green function, its value is not proportional to the enclosed mass but depends on the particular choice of the surface

$$\int_{\Sigma} d\Sigma \quad \mathbf{g}_{New.mech.} \cdot \hat{n} \propto M_{\Sigma} \quad (208)$$

Hence M_{Σ} is a mass-function depending on the surface Σ . Then we have to find the solution inside/outside the matter distribution by evaluating the integral quantities and imposing the boundary condition on the separation surface.

We have to note that, for any function of modulus $h(|\mathbf{x}|)$, it is

$$I = \int d^3\mathbf{x}' \mathcal{G}(\mathbf{x}, \mathbf{x}') h(\mathbf{x}') = -\frac{1}{4\pi} \int d|\mathbf{x}'| |\mathbf{x}'|^2 h(|\mathbf{x}'|) \int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \frac{\sin \theta' e^{-m\sqrt{|\mathbf{x}|^2 + |\mathbf{x}'|^2 - 2|\mathbf{x}||\mathbf{x}'| \cos \alpha}}}{\sqrt{|\mathbf{x}|^2 + |\mathbf{x}'|^2 - 2|\mathbf{x}||\mathbf{x}'| \cos \alpha}} \quad (209)$$

where $\cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$ and α is the angle between two vectors \mathbf{x} , \mathbf{x}' . In the spherically symmetric case we can choose $\theta = 0$ without losing generality (the symmetry of system is independent by the angle). By making the angular integration we get

$$I = -\frac{1}{2m|\mathbf{x}|} \int d|\mathbf{x}'| |\mathbf{x}'| h(|\mathbf{x}'|) \left[e^{-m||\mathbf{x}| - |\mathbf{x}'||} - e^{-m(|\mathbf{x}| + |\mathbf{x}'|)} \right] \quad (210)$$

An analogous relation is useful also for the Green function of Newtonian mechanics $|\mathbf{x} - \mathbf{x}'|^{-1}$

$$\int d^3\mathbf{x}' \frac{h(|\mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} = -\frac{2\pi}{|\mathbf{x}|} \int d|\mathbf{x}'| |\mathbf{x}'| \left[||\mathbf{x}| - |\mathbf{x}'|| - |\mathbf{x}| - |\mathbf{x}'| \right] h(|\mathbf{x}'|) \quad (211)$$

1. Solutions at $\mathcal{O}(2)$ - and $\mathcal{O}(3)$ -order

By supposing that $m^2 > 0$ (*i.e.* $\text{sign}[f'(0)] = -\text{sign}[f''(0)]$) (an analogous condition used in (184)), the Ricci scalar (194) is⁸

$$R^{(2)}(t, \mathbf{x}) = -\frac{3r_g}{\xi^3} \left[1 - e^{-m\xi} (1 + m\xi) \frac{\sinh m|\mathbf{x}|}{m|\mathbf{x}|} \right] \Theta(\xi - |\mathbf{x}|) - r_g m^2 F(\xi) \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \Theta(|\mathbf{x}| - \xi) \quad (212)$$

where we introduced a *shape function*

$$F(x) \doteq 3 \frac{mx \cosh mx - \sinh mx}{m^3 x^3} \quad (213)$$

The solutions of (196), (197) and (198), given the relations (210) and (211), respectively are

⁸ We have set for simplicity $f'(0) = 1$, otherwise we have to renormalize the coupling constant \mathcal{X} in the action (18).

$$\begin{aligned}
g_{tt}^{(2)}(t, \mathbf{x}) = & - r_g \left[\frac{3}{2\xi} + \frac{1}{m^2\xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} - \frac{e^{-m\xi}(1+m\xi)}{m^2\xi^3} \frac{\sinh m|\mathbf{x}|}{m|\mathbf{x}|} \right] \Theta(\xi - |\mathbf{x}|) \\
& - r_g \left[\frac{1}{|\mathbf{x}|} + \frac{F(\xi)}{3} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \right] \Theta(|\mathbf{x}| - \xi)
\end{aligned} \tag{214}$$

$$g_{ti}^{(3)}(t, \mathbf{x}) = 0 \tag{215}$$

$$\begin{aligned}
g_{ij}^{(2)}(t, \mathbf{x}) = & - r_g \left\{ \left[\frac{3}{2\xi} - \frac{5}{3m^2\xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} + \frac{(1+m\xi)e^{-m\xi}}{3m^2\xi^3} \left(2F(\mathbf{x}) + 3 \frac{\sinh m|\mathbf{x}|}{m|\mathbf{x}|} \right) \right] \Theta(\xi - |\mathbf{x}|) \right. \\
& \left. + \left[\frac{1}{|\mathbf{x}|} - \frac{2}{3m^2|\mathbf{x}|^3} - \frac{F(\xi)}{3} \left(\frac{1}{|\mathbf{x}|} - \frac{2}{m|\mathbf{x}|^2} - \frac{2}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} \right] \Theta(|\mathbf{x}| - \xi) \right\} \delta_{ij} \\
& - r_g \left\{ \left[\frac{2(1+m\xi)e^{-m\xi}}{m^2\xi^3} \left(\frac{\sinh m|\mathbf{x}|}{m|\mathbf{x}|} - F(\mathbf{x}) \right) \right] \Theta(\xi - |\mathbf{x}|) \right. \\
& \left. + \left[\frac{2}{m^2|\mathbf{x}|^3} - \frac{2F(\xi)}{3} \left(\frac{1}{|\mathbf{x}|} + \frac{3}{m|\mathbf{x}|^2} + \frac{3}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} \right] \Theta(|\mathbf{x}| - \xi) \right\} \frac{x_i x_j}{|\mathbf{x}|^2}
\end{aligned} \tag{216}$$

We note that the corrections to GR behavior are ruled by $\mathcal{G}(\mathbf{x}, \mathbf{x}')$. If we perform a Taylor expansion for $m|\mathbf{x}| \ll 1$, we have

$$\frac{\sinh m|\mathbf{x}|}{m|\mathbf{x}|} \simeq 1 + \frac{m^2|\mathbf{x}|^2}{6} + \dots \tag{217}$$

For fixed values of the distance $|\mathbf{x}|$, the solutions $g_{tt}^{(2)}$ and $g_{ij}^{(2)}$ depend on the value of the radius ξ , then the Gauss theorem does not hold also if the Bianchi identities hold, as already said above [12]. In other words, since the Green function does not scale as the inverse of distance but has also an exponential behavior, the Gauss theorem is not verified. We can affirm: *the potential does not depend only on the total mass but also on the mass-distribution in the space*. We can write

$$\lim_{\xi \rightarrow \infty} F(m\xi) = \infty \tag{218}$$

obviously the limit of ξ has to be interpreted up to the maximal value where the generic position $|\mathbf{x}|$ in the space is fixed. If we consider the limit $\xi \rightarrow 0$ (the point-like source limit), we obtain

$$\lim_{\xi \rightarrow 0} F(m\xi) = 1 \tag{219}$$

By introducing three metric potentials $\Phi(\mathbf{x})$, $\Psi(\mathbf{x})$ and $\Lambda(\mathbf{x})$ (the dimension is the inverse of length) we can rewrite (214) and (216) as follows

$$\begin{cases} g_{tt}^{(2)}(t, \mathbf{x}) = r_g \Phi(\mathbf{x}) \\ g_{ij}^{(2)}(t, \mathbf{x}) = r_g \Psi(\mathbf{x}) \delta_{ij} + r_g \Lambda(\mathbf{x}) \frac{x_i x_j}{|\mathbf{x}|^2} \end{cases} \tag{220}$$

and with a fourth function, $\Xi(\mathbf{x})$, (the dimension is the cubic inverse of length) the Ricci scalar (212) is

$$R^{(2)}(t, \mathbf{x}) = r_g \Xi(\mathbf{x}) \tag{221}$$

The spatial behavior (212) is shown in FIG. 1. The metric potentials are shown in FIGs. 2, 3 and 4. It is interesting to note as the function Φ assumes smaller value of its equivalent in GR, then in terms of gravitational attraction we have a potential well more deep. A such scheme can be interpretable or assuming a variation of the gravitational constant G or requiring that there is a central greater mass. These two affirmations are compatible on the one hand with the tensor-scalar theories (in the which we have a scaling of gravitational constant) and on the other hand with the theory of GR plus the hypothesis of the existence of the dark matter. In particular, if the mass distribution takes a bigger volume, the potential increases and vice versa.

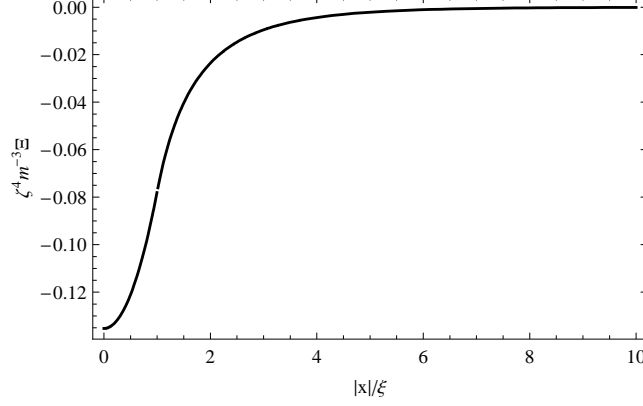


FIG. 1: Plot of dimensionless function $\zeta^4 m^{-3} \Xi$ for $\zeta \doteq m\xi = 0.5$ representing the spatial behavior of Ricci scalar at second order. In GR we would have $\Xi(\mathbf{x}) = \frac{3}{\xi^3} \Theta(\xi - |\mathbf{x}|)$.

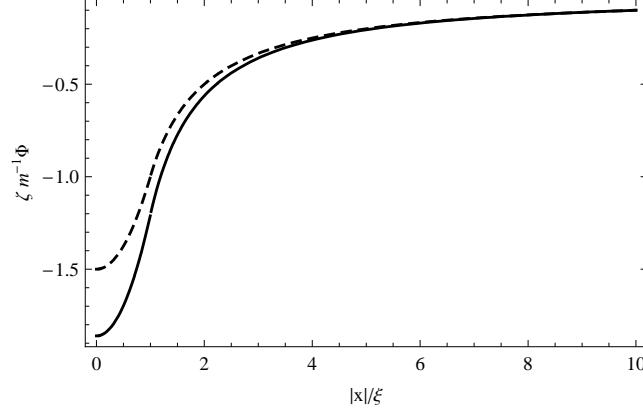


FIG. 2: Plot of metric potential $\zeta m^{-1} \Phi$ vs distance from central mass with $\zeta \doteq m\xi = 0.5$. The dashed line is the GR behavior:

$$\Phi = - \left[\frac{3}{2\xi} - \frac{|\mathbf{x}|^2}{2\xi^3} \right] \Theta(\xi - \mathbf{x}) - \frac{\Theta(\mathbf{x} - \xi)}{|\mathbf{x}|}.$$

In the limit of point-like source, *i.e.* $\lim_{\xi \rightarrow 0} \frac{3M}{3\pi\xi^3} \Theta(\xi - |\mathbf{x}|) = M\delta(\mathbf{x})$, we get

$$\left\{ \begin{array}{l} R^{(2)}(t, \mathbf{x}) = -r_g m^2 \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \\ g_{tt}^{(2)}(t, \mathbf{x}) = -r_g \left(\frac{1}{|\mathbf{x}|} + \frac{1}{3} \frac{e^{-m|\mathbf{x}|}}{|\mathbf{x}|} \right) \\ g_{ij}^{(2)}(t, \mathbf{x}) = -r_g \left\{ \frac{1}{|\mathbf{x}|} - \frac{2}{3m^2|\mathbf{x}|^3} - \frac{1}{3} \left(\frac{1}{|\mathbf{x}|} - \frac{2}{m|\mathbf{x}|^2} - \frac{2}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} \right\} \delta_{ij} \\ \quad - r_g \left[\frac{2}{m^2|\mathbf{x}|^3} - \frac{2}{3} \left(\frac{1}{|\mathbf{x}|} + \frac{3}{m|\mathbf{x}|^2} + \frac{3}{m^2|\mathbf{x}|^3} \right) e^{-m|\mathbf{x}|} \right] \frac{x_i x_j}{|\mathbf{x}|^2} \end{array} \right. \quad (222)$$

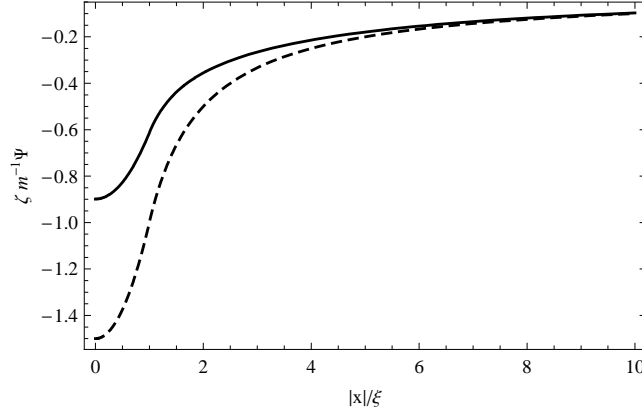


FIG. 3: Plot of metric potential $\zeta m^{-1} \Psi$ vs distance from central mass with $\zeta \doteq m\xi = 0.5$. The dashed line is the GR behavior (similar to metric potential Φ).

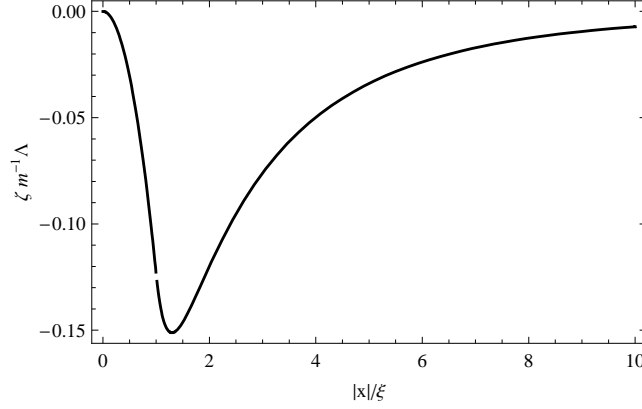


FIG. 4: Plot of metric potential $\zeta m^{-1} \Lambda$ vs distance from central mass with $\zeta \doteq m\xi = 0.5$. In GR such a behavior is missing.

An important remark has to be done at this point. Now we can check the compatibility of $f(R)$ -gravity with respect to GR in the limit ($f(R) \rightarrow R$). Since the Gauss theorem is not verified for $f(R)$ -gravity, while the relations (69) satisfy it, we have to consider the relations (222) and not (214), (216). After making the limit $f(R) \rightarrow R$ we have

$$\begin{cases} R^{(2)}(t, \mathbf{x}) = 0 \\ g_{tt}^{(2)}(t, \mathbf{x}) = -\frac{r_g}{|\mathbf{x}|} \\ g_{ij}^{(2)}(t, \mathbf{x}) = -\frac{r_g}{|\mathbf{x}|} \delta_{ij} \end{cases} \quad (223)$$

which suggest that the $f(R)$ -gravity is compatible with respect to GR. It is interesting to note that also in the case of extended spherically symmetric distribution of matter, when we perform the limit $f(R) \rightarrow R$, the solutions (214) and (216) directly converge (in the vacuum) to solutions (223), showing the validity of Gauss theorem in GR.

Another important consideration is about the asymptotic behavior of $f(R)$ -gravity with respect to GR. In fact, increasing the distance from the central mass, the gravitational field should converge to that of GR. Such a convergence is the standard consequence of the spherically symmetry of the source with asymptotically flat boundary conditions. In FIG. 1, we report the spatial behavior of Ricci scalar (212) approximating asymptotically the given value in GR. In fact supposing $f(R)$ - gravity, the Ricci scalar acquires dynamics, and in the Newtonian limit, we find a characteristic scale length (m^{-1}) related to the scalar massive mode. Only for distances larger than m^{-1} , we recover the outcome of GR, that is $R = 0$. The metric potentials are shown in FIGs. 2, 3 and 4.

To conclude this section, we show in FIG. 5 the comparison between gravitational forces induced in GR and in $f(R)$ -gravity considered in the Newtonian limit. Obviously also considering forces, we could obtained an intensity

different than that in GR.

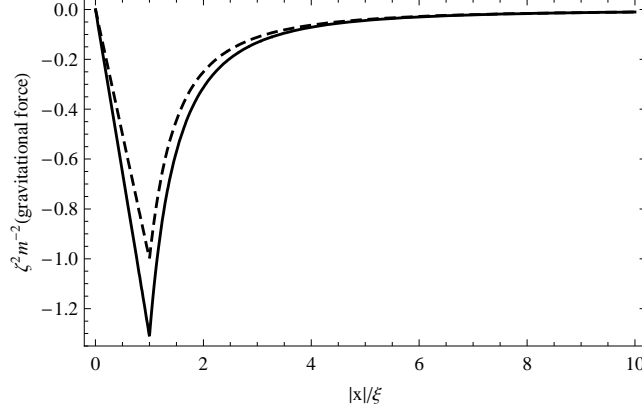


FIG. 5: Comparison between gravitational forces induced by GR and $f(R)$ -gravity with $\zeta \doteq m\xi = 0.5$. The dashed line is the GR behavior.

2. The Newtonian Limit of $f(R)$ -gravity in oscillating regime

If we consider $m^2 < 0$ (i.e. $\text{sign}[f'(0)] = \text{sign}[f''(0)]$) from (204) we can choose the "oscillating" Green function

$$\mathcal{G}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{\cos m|\mathbf{x} - \mathbf{x}'| + \sin m|\mathbf{x} - \mathbf{x}'|}{|\mathbf{x} - \mathbf{x}'|} \quad (224)$$

The Ricci scalar (194) and the tt -component of $g_{\mu\nu}$ at $\mathcal{O}(2)$ order (196) become

$$R^{(2)}(t, \mathbf{x}) = -\frac{6r_g}{\xi^3} \left[1 - H(\xi) \frac{\sin m|\mathbf{x}|}{m|\mathbf{x}|} \right] \Theta(\xi - |\mathbf{x}|) - 2r_g m^2 G(\xi) \frac{\cos m|\mathbf{x}| + \sin m|\mathbf{x}|}{|\mathbf{x}|} \Theta(|\mathbf{x}| - \xi) \quad (225)$$

$$\begin{aligned} g_{tt}^{(2)}(t, \mathbf{x}) = & -r_g \left[\frac{3}{2\xi} - \frac{2}{m^2 \xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} + \frac{2H(\xi) \sin m|\mathbf{x}|}{m^2 \xi^3 m|\mathbf{x}|} \right] \Theta(\xi - |\mathbf{x}|) \\ & - r_g \left[\frac{1}{|\mathbf{x}|} - \frac{2G(\xi) \cos m|\mathbf{x}| + \sin m|\mathbf{x}|}{3|\mathbf{x}|} \right] \Theta(|\mathbf{x}| - \xi) \end{aligned} \quad (226)$$

where we introduced two new *shape functions*

$$G(x) \doteq 3 \frac{mx \cos mx - \sin mx}{m^3 x^3}, \quad H(x) \doteq (1 - mx) \cos mx + (1 + mx) \sin mx \quad (227)$$

with the properties $\lim_{\xi \rightarrow 0} G(\xi) = -1$ and $\lim_{\xi \rightarrow 0} H(\xi) = 1$. Since we have an oscillating Green function which is not asymptotically zero, the "gravitational potentials" (226) at infinity are zero a part a possible constant value ($\lim_{a \rightarrow \infty} 2r_g m G(\xi) (\sin ma - \cos ma)$).

The spatial behavior of Ricci scalar (225) and metric component (226) are shown in FIGs. 6 and 7. The previous considerations hold also for the solutions (225) - (226). The only difference is that now we have oscillating behaviors instead of exponential behaviors.

The correction term to the Newtonian potential in the external solution can be interpreted as the Fourier transform of the matter density $\rho(\mathbf{x})$. In fact, we have

$$\int \frac{d^3 \mathbf{x}'}{(2\pi)^3} \rho(\mathbf{x}') e^{-i\mathbf{k} \cdot \mathbf{x}'} = -\frac{MG(|\mathbf{k}|\xi)}{(2\pi)^3} \quad (228)$$

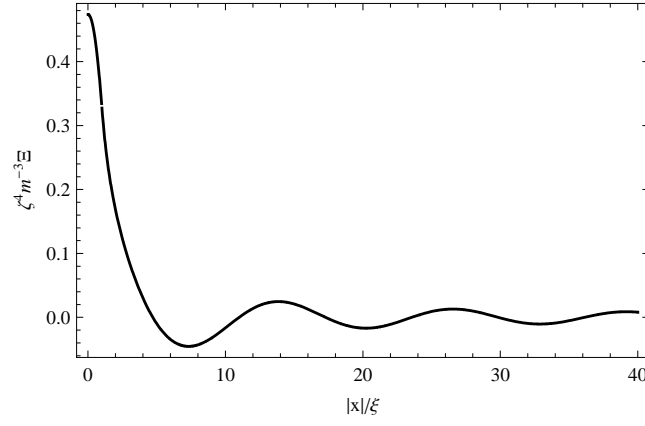


FIG. 6: Plot of dimensionless function $\zeta^4 m^{-3} \Xi$ with $\zeta \doteq m\xi = .5$ representing the spatial behavior of Ricci scalar at second order in the oscillating case.

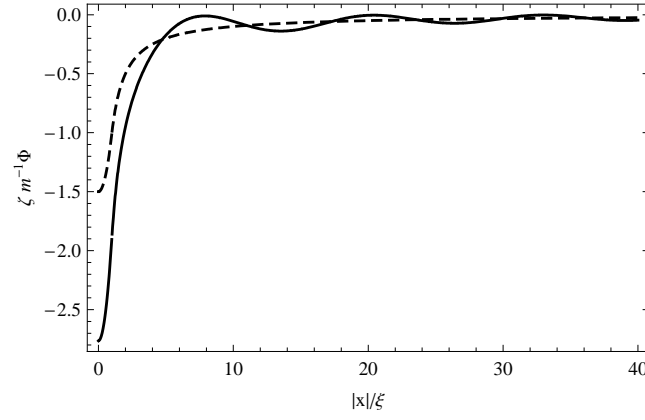


FIG. 7: Plot of metric potential $\zeta m^{-1} \Phi$ vs distance from central mass with the choice $\zeta \doteq m\xi = 0.5$ in the oscillating case. The dashed line is the GR behavior.

and in the limit of point-like source

$$\lim_{\xi \rightarrow 0} \int \frac{d^3 \mathbf{x}'}{(2\pi)^3} \rho(\mathbf{x}') e^{-i\mathbf{k} \cdot \mathbf{x}'} = \frac{M}{(2\pi)^3} \quad (229)$$

Also in this case we show in FIG. 8 the comparison between gravitational forces induced in GR and in $f(R)$ -gravity in the Newtonian limit. Obviously also in this last case we obtained a different force with respect to GR.

3. Solutions at $\mathcal{O}(4)$ -order

The metric potentials and the function $\Xi(\mathbf{x})$, respectively defined in (220) and (221), satisfy the following properties with respect to derivative of coordinate l -th⁹ in the matter

⁹ We remember that $|\mathbf{x}|_{,l} = |\mathbf{x}|^{-1} x_l$.

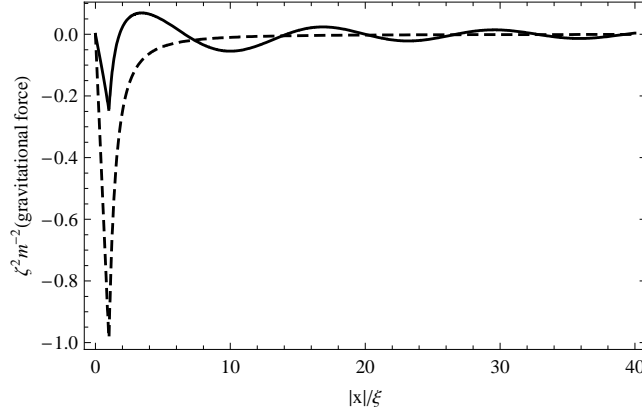


FIG. 8: Comparison between gravitational forces induced by GR and $f(R)$ -gravity with $\zeta \doteq m\xi = 0.5$ in the oscillating case. The dashed line is the GR behavior.

$$\left\{ \begin{array}{l} \Xi_{,l}(\mathbf{x}) = \frac{m^2(1+m\xi)}{\xi^3} e^{-m\xi} F(\mathbf{x}) x_l = \Xi_0(\mathbf{x}) x_l \\ \Phi_{,l}(\mathbf{x}) = \left[\frac{1}{\xi^3} + \frac{(1+m\xi)}{3\xi^3} e^{-m\xi} F(\mathbf{x}) \right] x_l = \Phi_0(\mathbf{x}) x_l \\ \Psi_{,l}(\mathbf{x}) = \left[\frac{1}{\xi^3} - \frac{(1+m\xi)}{\xi^3} e^{-m\xi} \frac{(m^2|\mathbf{x}|^2+6) \sinh m|\mathbf{x}| + m|\mathbf{x}|(m^2|\mathbf{x}|^2-6) \cosh m|\mathbf{x}|}{m^5|\mathbf{x}|^5} \right] x_l = \Psi_0(\mathbf{x}) x_l \\ \Lambda_{,l}(\mathbf{x}) = \frac{2(1+m\xi)}{\xi^3} e^{-m\xi} \frac{(4m^2|\mathbf{x}|^2+9) \sinh m|\mathbf{x}| - m|\mathbf{x}|(m^2|\mathbf{x}|^2+9) \cosh m|\mathbf{x}|}{m^5|\mathbf{x}|^5} x_l = \Lambda_0(\mathbf{x}) x_l \\ \Xi_{,ln}(\mathbf{x}) = \Xi_0(\mathbf{x}) \delta_{ln} + \frac{3(1+m\xi)}{\xi^3} e^{-m\xi} \frac{(m^2|\mathbf{x}|^2+3) \sinh m|\mathbf{x}| - 3m|\mathbf{x}| \cosh m|\mathbf{x}|}{m|\mathbf{x}|^5} x_l x_n = \Xi_0(\mathbf{x}) \delta_{ln} + \Xi_1(\mathbf{x}) x_l x_n \\ \Phi_{,ln}(\mathbf{x}) = \Phi_0(\mathbf{x}) \delta_{ln} + \frac{(1+m\xi)}{\xi^3} e^{-m\xi} \frac{(m^2|\mathbf{x}|^2+3) \sinh m|\mathbf{x}| - 3m|\mathbf{x}| \cosh m|\mathbf{x}|}{m^3|\mathbf{x}|^5} x_l x_n = \Phi_0(\mathbf{x}) \delta_{ln} + \Phi_1(\mathbf{x}) x_l x_n \end{array} \right. \quad (230)$$

and in the vacuum

$$\left\{ \begin{array}{l} \Xi_{,l}(\mathbf{x}) = \frac{m^2(m|\mathbf{x}|+1)}{|\mathbf{x}|^3} F(\xi) e^{-m|\mathbf{x}|} x_l = \Xi_0(\mathbf{x}) x_l \\ \Phi_{,l}(\mathbf{x}) = \left[\frac{1}{|\mathbf{x}|^3} + \frac{m|\mathbf{x}|+1}{|\mathbf{x}|^3} \frac{F(\xi) e^{-m|\mathbf{x}|}}{3} \right] x_l = \Phi_0(\mathbf{x}) x_l \\ \Psi_{,l}(\mathbf{x}) = \left[\frac{m^2|\mathbf{x}|^2-2}{m^2|\mathbf{x}|^5} - \frac{m^3|\mathbf{x}|^3-m^2|\mathbf{x}|^2-6m|\mathbf{x}|-6}{m^2|\mathbf{x}|^5} \frac{F(\xi) e^{-m|\mathbf{x}|}}{3} \right] x_l = \Psi_0(\mathbf{x}) x_l \\ \Lambda_{,l}(\mathbf{x}) = \left[\frac{6}{m^2|\mathbf{x}|^5} - \frac{m^3|\mathbf{x}|^3+4m^2|\mathbf{x}|^2+9m|\mathbf{x}|+9}{m^2|\mathbf{x}|^5} \frac{2F(\xi) e^{-m|\mathbf{x}|}}{3} \right] x_l = \Lambda_0(\mathbf{x}) x_l \\ \Xi_{,ln}(\mathbf{x}) = \Xi_0(\mathbf{x}) \delta_{ln} - \frac{m^2(m^2|\mathbf{x}|^2+3m|\mathbf{x}|+3)}{|\mathbf{x}|^5} F(\xi) e^{-m|\mathbf{x}|} x_l x_n = \Xi_0(\mathbf{x}) \delta_{ln} + \Xi_1(\mathbf{x}) x_l x_n \\ \Phi_{,ln}(\mathbf{x}) = \Phi_0(\mathbf{x}) \delta_{ln} - \left[\frac{3}{|\mathbf{x}|^5} + \frac{m^2|\mathbf{x}|^2+3m|\mathbf{x}|+3}{|\mathbf{x}|^5} \frac{F(\xi) e^{-m|\mathbf{x}|}}{3} \right] x_l x_n = \Phi_0(\mathbf{x}) \delta_{ln} + \Phi_1(\mathbf{x}) x_l x_n \end{array} \right. \quad (231)$$

Obviously when we consider the physics in the matter or in the vacuum we have to choose the "right" quantities $\Xi_0(\mathbf{x})$, $\Xi_1(\mathbf{x})$, $\Phi_0(\mathbf{x})$, $\Phi_1(\mathbf{x})$, $\Psi_0(\mathbf{x})$, $\Lambda_0(\mathbf{x})$.

The expression of Ricci scalar at fourth order (199) is

$$R^{(4)}(t, \mathbf{x}) = \frac{r_g^2}{2m|\mathbf{x}|} \int_0^\infty d|\mathbf{x}'||\mathbf{x}'| \left\{ e^{-m||\mathbf{x}|-|\mathbf{x}'||} - e^{-m(|\mathbf{x}|+|\mathbf{x}'|)} \right\} \left\{ \frac{m^4}{\mu^4} \left[\Xi(\mathbf{x}')^2 + \frac{|\mathbf{x}'|^2}{m^2} \Xi_0(\mathbf{x}')^2 \right] \right. \\ \left. + \left[3\Lambda(\mathbf{x}') + \frac{\Phi_0(\mathbf{x}') - \Psi_0(\mathbf{x}') + \Lambda_0(\mathbf{x}')}{2} |\mathbf{x}'|^2 \right] \Xi_0(\mathbf{x}') + m^2 \Psi(\mathbf{x}') \Xi(\mathbf{x}') + \Xi_1(\mathbf{x}') \Lambda(\mathbf{x}') |\mathbf{x}'|^2 \right\} \quad (232)$$

where we note two contributions. The first one still depends on the quadratic term ($\propto R^2$) in the action (18), while the second one is related to the cubic term ($\propto R^3$). By introducing the two functions $\Xi_I(\mathbf{x})$ and $\Xi_{II}(\mathbf{x})$, Eq. (232) can be rewritten as follows

$$R^{(4)}(t, \mathbf{x}) = r_g^2 \left[\Xi_I(\mathbf{x}) + \frac{m^4}{\mu^4} \Xi_{II}(\mathbf{x}) \right] \quad (233)$$

An analogous situation is found for the tt -component of metric tensor at fourth order. In fact Eq. (201) becomes

$$g_{tt}^{(4)}(t, \mathbf{x}) = \frac{r_g \mathcal{X}}{|\mathbf{x}|} \int_0^\xi d|\mathbf{x}'||\mathbf{x}'| \left\{ ||\mathbf{x}| - |\mathbf{x}'|| - |\mathbf{x}| - |\mathbf{x}'| \right\} \rho(\mathbf{x}') \Phi(\mathbf{x}') \\ + \frac{r_g^2}{|\mathbf{x}|} \int_0^\infty d|\mathbf{x}'||\mathbf{x}'| \left\{ ||\mathbf{x}| - |\mathbf{x}'|| - |\mathbf{x}| - |\mathbf{x}'| \right\} \left\{ \frac{1}{2} \left[\Xi_I(\mathbf{x}') + \Phi(\mathbf{x}') \Xi(\mathbf{x}') + \Phi_0(\mathbf{x}')^2 |\mathbf{x}'|^2 \right] \right. \\ \left. - \frac{1}{2} \left[\left(3\Psi(\mathbf{x}') + \Lambda(\mathbf{x}') \right) \Phi_0(\mathbf{x}') + \left(\Psi(\mathbf{x}') + \Lambda(\mathbf{x}') \right) \Phi_1(\mathbf{x}') |\mathbf{x}'|^2 \right] \right\} \\ - \frac{r_g^2}{3m^2|\mathbf{x}|} \int_0^\infty d|\mathbf{x}'||\mathbf{x}'| \left\{ ||\mathbf{x}| - |\mathbf{x}'|| - |\mathbf{x}| - |\mathbf{x}'| \right\} \left\{ \frac{\Xi(\mathbf{x}')^2}{4} + \Delta_{\mathbf{x}'} \Xi_I(\mathbf{x}') + \Xi_1(\mathbf{x}') \Lambda(\mathbf{x}') |\mathbf{x}'|^2 \right. \\ \left. + \frac{\Xi_0(\mathbf{x}')}{2} \left[6\Lambda(\mathbf{x}') + \left(5\Psi_0(\mathbf{x}') + 3\Lambda_0(\mathbf{x}') \right) |\mathbf{x}'|^2 \right] \right. \\ \left. + \frac{\Xi(\mathbf{x}')}{2} \left[2m^2 \left(\Phi(\mathbf{x}') + \Psi(\mathbf{x}') \right) - 3\Phi_0(\mathbf{x}') - \Phi_1 |\mathbf{x}'|^2 \right] \right\} \\ + \frac{m^4}{\mu^4} \frac{r_g^2}{|\mathbf{x}|} \int_0^\infty d|\mathbf{x}'||\mathbf{x}'| \left\{ ||\mathbf{x}| - |\mathbf{x}'|| - |\mathbf{x}| - |\mathbf{x}'| \right\} \left\{ \frac{\Xi_{II}(\mathbf{x}')}{2} - \frac{m^2 \Xi(\mathbf{x}')^2 + |\mathbf{x}'|^2 \Xi_0(\mathbf{x}')^2}{3m^4} \right. \\ \left. - \frac{\Delta_{\mathbf{x}'} \Xi_{II}(\mathbf{x}')}{3m^2} \right\} \quad (234)$$

and by introducing other new functions $\Phi_I(\mathbf{x})$, $\Phi_{II}(\mathbf{x})$, finally we have

$$g_{tt}^{(4)}(t, \mathbf{x}) = r_g^2 \left[\Phi_I(\mathbf{x}) + \frac{m^4}{\mu^4} \Phi_{II}(\mathbf{x}) \right] \quad (235)$$

It is useful to note that we have generally four contributions to $g_{tt}^{(4)}$ in (234). The first one is induced by the non-linearity of the metric tensor even in static spherically symmetric case. The product $\rho(\mathbf{x})\Phi(\mathbf{x})$ is non-zero only in the matter but contributes to determination of tt -component in any point of the space. The second one takes into account the contribution induced by the solution of previous order for the determination of the tt -component of the Ricci tensor at fourth order. These first two terms are present also in GR. While the second two terms are derived from the modification of the theory. In fact the third contribution depends on the addition of the quadratic term ($\propto R^2$) in the action and finally the fourth one from the addition of the cubic term ($\propto R^3$).

The choice of free parameter μ , which is linked to the third derivative of $f(R)$, is a crucial point in both the expressions (233) and (235) to obtain the right behavior. From the mathematical interpretation of Newtonian limit one has $|f'''(0)| < |f''(0)|$ and if $\mu^4 > 0$ (*i.e.* $\text{sign}[f'(0)] = -\text{sign}[f'''(0)]$, otherwise μ^4 is not a length) we have $m^4/\mu^4 = |f'''(0)|/3f''(0)^2$, so we find the constraint $0 < m^4/\mu^4 < 1$. In FIG. 9, we report the spatial behavior

of (233) in the matter and in the vacuum, ($0.3 \leq m^4/\mu^4 \leq 0.9$), showing that far from the source we obtain a spacetime with a vanishing scalar curvature. At Newtonian level, the Ricci scalar $R^{(2)}$ (212) is negative defined while at post-Newtonian limit it is positive defined.

In FIG. (10), we report the time-time component of metric tensor $g_{tt}^{(4)}$ on the same interval of values of m^4/μ^4 , although the behavior is quite insensitive to changes induced by the contributions of the cubic term in the Lagrangian. Besides we can observe an important analogy with respect the results of GR. In both cases, we have a potential barrier, but for $f(R)$ -gravity it is higher (as in the Newtonian limit we found a deeper potential well).

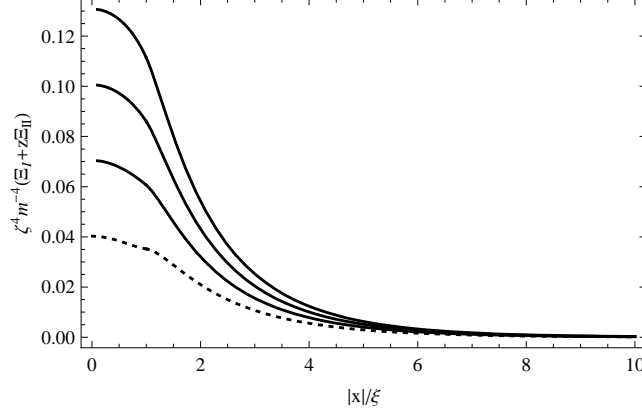


FIG. 9: Plot of dimensionless function $\zeta^4 m^{-4}(\Xi_I + z \Xi_{II})$, representing the Ricci scalar at fourth order, where $z = m^4/\mu^4$ and $\zeta \doteq m\xi = 0.5$. The spatial behavior is shown for $0.3 \leq z \leq 0.9$ (solid lines) while the dotted line corresponds to $R - \frac{1}{6m^2}R^2$ -theory.

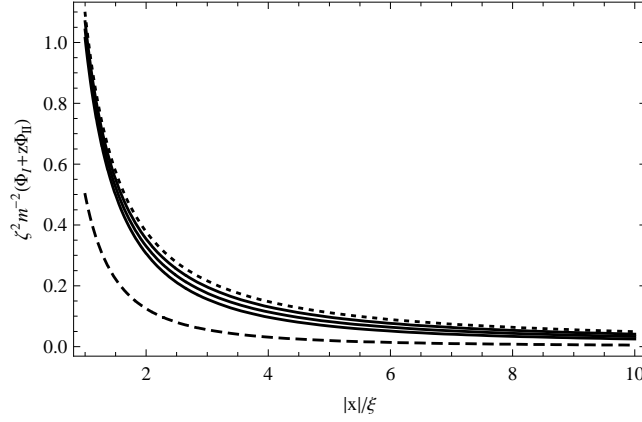


FIG. 10: Plot of dimensionless function $\zeta^2 m^{-2}(\Phi_I + z \Phi_{II})$ (solid lines) where $z = m^4/\mu^4$ and of function $1/2|\mathbf{x}|^2$ (dashed line). For $z = 0$ (dotted line) we have the behavior of $R - \frac{1}{6m^2}R^2$ -theory. The solid lines are obtained for $0.3 \leq z \leq 0.9$ and for $\zeta = m\xi = 0.5$.

4. Solutions from isotropic to standard coordinates

The metric solutions that we have found are expressed in isotropic coordinates and often, for spherically symmetric problems, they are conveniently rewritten in standard coordinates (the standard form in which we write the Schwarzschild solution). Here the relativistic invariant of metric (189) is

$$ds^2 = \left[1 + r_g \Phi(\mathbf{x}) + r_g^2 \left(\Phi_I(\mathbf{x}) + \frac{m^4}{\mu^4} \Phi_{II}(\mathbf{x}) \right) \right] dt^2 - \left[1 - r_g \Psi(\mathbf{x}) \right] |d\mathbf{x}|^2 + r_g \Lambda(\mathbf{x}) \frac{(\mathbf{x} \cdot d\mathbf{x})^2}{|\mathbf{x}|^2} \quad (236)$$

From spherically symmetric form of (236), it is convenient to replace it with Eq. (42) by transformation (34). The proper time interval (236) then becomes

$$ds^2 = \left[1 + r_g \Phi(r) + r_g^2 \left(\Phi_I(r) + \frac{m^4}{\mu^4} \Phi_{II}(r) \right) \right] dt^2 - \left[1 - r_g \left(\Psi(r) + \Lambda(r) \right) \right] dr^2 - \left[1 - r_g \Psi(r) \right] r^2 d\Omega \quad (237)$$

To get the metric in the standard form, we need to impose a radial coordinate transformation

$$\left[1 - r_g \Psi(r) \right] r^2 = \tilde{r}^2 \quad (238)$$

and we have a new set of coordinates $\{\tilde{r}, \theta, \phi\}$. The metric (237) becomes

$$ds^2 = \left[1 + r_g \tilde{\Phi}(\tilde{r}) + r_g^2 \left(\tilde{\Phi}_I(\tilde{r}) + \frac{m^4}{\mu^4} \tilde{\Phi}_{II}(\tilde{r}) \right) \right] dt^2 - \left[1 - r_g \left(\tilde{\Psi}(\tilde{r}) + \tilde{\Lambda}(\tilde{r}) \right) \right] \left(\frac{dr}{d\tilde{r}} \right)^2 d\tilde{r}^2 - \tilde{r}^2 d\Omega \quad (239)$$

The explicit expression of (239) is not displayed because the equation (238) cannot be solved algebraically. However, this technical problem is overcome with the help of numerical methods when we are interested to test experimentally the theory.

VIII. THE NEWTONIAN LIMIT OF QUADRATIC GRAVITY: $f(X, Y) = a_1 R + a_2 R^2 + a_3 R_{\alpha\beta} R^{\alpha\beta}$

Since terms resulting from R^n with $n \geq 3$ *do not contribute* in the Newtonian limit, as we have seen previously, we provide explicit solutions for different types of Lagrangians generated by a primitive Lagrangian, the so-called *Quadratic Lagrangian* of the form [58]

$$f(X, Y) = a_1 R + a_2 R^2 + a_3 R_{\alpha\beta} R^{\alpha\beta} \quad (240)$$

where a_1 , a_2 and a_3 are arbitrary constants¹⁰. Such a Lagrangian belongs to the general class of FOG (18). The field equations coming from (240) are

$$\begin{cases} H_{\mu\nu} = (a_1 + 2a_2 R) R_{\mu\nu} - \frac{a_1 R + a_2 R^2 + a_3 R_{\alpha\beta} R^{\alpha\beta}}{2} g_{\mu\nu} - 2a_2 R_{;\mu\nu} + 2a_2 g_{\mu\nu} \square R + 2a_3 R_{\mu}{}^{\alpha} R_{\alpha\nu} - 2a_3 R^{\alpha}{}_{(\mu;\nu)\alpha} \\ \quad + a_3 \square R_{\mu\nu} + a_3 R_{\alpha\beta}{}^{;\alpha\beta} g_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\ H = -a_1 R + 2(3a_2 + a_3) \square R = \mathcal{X} T \end{cases} \quad (241)$$

Till now the solutions of field equations (in general Eqs. (21)) are found by considering the trace Eq. (22) as the dynamical equation for the Ricci scalar. This approach allows to study a set of second order differential equations while the FOG equations are intrinsically fourth order differential equations in metric formalism. By using the trace equation (second line of (241)) or the definition of the Ricci scalar (5) and substituting them into the field equations (first line of (241)), we have a set of fourth order partial differential equations. In this section, we will study FOG field equations.

If we introduce the gravitational potentials in the isotropic metric (41) by the quantities Φ and Ψ linked to $g_{tt}^{(2)}$ and $g_{ij}^{(2)}$

$$ds^2 = \left[1 + 2\Phi \right] dt^2 - \left[1 - 2\Psi \right] \delta_{ij} dx^i dx^j \quad (242)$$

¹⁰ Note that the physical dimensions of the constants are $[a_2] = [a_3] = \text{length}^2$ and $[a_1] = \text{length}^0$.

and by using the paradigm of Newtonian and post-Newtonian developments, we can investigate, without assuming the harmonic gauge condition, field Eqs.(241) for the *Quadratic Lagrangian* (240) at Newtonian order, that is

$$\begin{cases} 2a_1\Delta\Psi - 2(4a_2 + a_3)\Delta^2\Psi + 2(2a_2 + a_3)\Delta^2\Phi = \mathcal{X}\rho \\ \Delta\left[a_1(\Psi - \Phi) + (4a_2 + a_3)\Delta\Phi - (8a_2 + 3a_3)\Delta\Psi\right]\delta_{ij} - \left[a_1(\Psi - \Phi) + (4a_2 + a_3)\Delta\Phi - (8a_2 + 3a_3)\Delta\Psi\right]_{,ij} = 0 \end{cases} \quad (243)$$

By introducing two new auxiliary functions ($\tilde{\Phi}$ and $\tilde{\Psi}$), Eqs.(243) become

$$\begin{cases} \frac{2a_2}{2a_2+a_3}\Delta^2\tilde{\Psi} - \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta^2\tilde{\Phi} - \frac{4a_2+a_3}{2a_2+a_3}\Delta\tilde{\Phi} - \frac{a_1}{2a_2+a_3}\Delta\tilde{\Psi} = \mathcal{X}\rho \\ \Delta\left[\tilde{\Phi} + \Delta\tilde{\Psi}\right]\delta_{ij} - \left[\tilde{\Phi} + \Delta\tilde{\Psi}\right]_{,ij} = 0 \end{cases} \quad (244)$$

where $\tilde{\Phi}$ and $\tilde{\Psi}$ are linked to Φ and Ψ via the transformations

$$\begin{cases} \Phi = -\frac{(8a_2+3a_3)\tilde{\Phi}+a_1\tilde{\Psi}}{2a_1(2a_2+a_3)} \\ \Psi = -\frac{(4a_2+a_3)\tilde{\Phi}+a_1\tilde{\Psi}}{2a_1(2a_2+a_3)} \end{cases} \quad (245)$$

Obviously we must require $a_1(2a_2 + a_3) \neq 0$, which is the determinant of the transformations (245). Let us introduce the new function \mathfrak{J} defined as follows

$$\mathfrak{J} \doteq \tilde{\Phi} + \Delta\tilde{\Psi} \quad (246)$$

At this point, we can use the new function Ξ to uncouple the system (243). With the choice $\tilde{\Phi} = \mathfrak{J} - \Delta\tilde{\Psi}$, it is possible to rewrite Eqs. (243) as follows

$$\begin{cases} \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta^3\tilde{\Psi} + \frac{6a_2+a_3}{2a_2+a_3}\Delta^2\tilde{\Psi} - \frac{a_1}{2a_2+a_3}\Delta\tilde{\Psi} = \mathcal{X}\rho + \tau \\ \Delta\mathfrak{J}\delta_{ij} - \mathfrak{J}_{,ij} = 0 \end{cases} \quad (247)$$

where $\tau \doteq \frac{4a_2+a_3}{2a_2+a_3}\Delta\mathfrak{J} + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta^2\mathfrak{J}$. We are interested in the solution of (244) in terms of the Green function $\mathfrak{T}(\mathbf{x}, \mathbf{x}')$ of field operator $\frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta^3 + \frac{6a_2+a_3}{2a_2+a_3}\Delta^2 - \frac{a_1}{2a_2+a_3}\Delta$. Then Eqs.(243) are equivalent to

$$\begin{cases} \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta^3\mathfrak{T}(\mathbf{x}, \mathbf{x}') + \frac{6a_2+a_3}{2a_2+a_3}\Delta^2\mathfrak{T}(\mathbf{x}, \mathbf{x}') - \frac{a_1}{2a_2+a_3}\Delta\mathfrak{T}(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \\ \Delta\mathfrak{J}(\mathbf{x})\delta_{ij} - \mathfrak{J}(\mathbf{x})_{,ij} = 0 \end{cases} \quad (248)$$

The general solutions of Eqs.(244) for $\Phi(\mathbf{x})$ and $\Psi(\mathbf{x})$, in terms of the Green function $\mathfrak{T}(\mathbf{x}, \mathbf{x}')$ and the function $\mathfrak{J}(\mathbf{x})$, are

$$\begin{cases} \Phi(\mathbf{x}) = \frac{(8a_2+3a_3)\Delta_{\mathbf{x}}-a_1}{2a_1(2a_2+a_3)} \int d^3\mathbf{x}' \mathfrak{T}(\mathbf{x}, \mathbf{x}') \left[\mathcal{X}\rho(\mathbf{x}') + \frac{4a_2+a_3}{2a_2+a_3}\Delta_{\mathbf{x}'}\mathfrak{J}(\mathbf{x}') + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta_{\mathbf{x}'}^2\mathfrak{J}(\mathbf{x}') \right] - \frac{8a_2+3a_3}{2a_1(2a_2+a_3)}\mathfrak{J}(\mathbf{x}) \\ \Psi(\mathbf{x}) = \frac{(4a_2+a_3)\Delta_{\mathbf{x}}-a_1}{2a_1(2a_2+a_3)} \int d^3\mathbf{x}' \mathfrak{T}(\mathbf{x}, \mathbf{x}') \left[\mathcal{X}\rho(\mathbf{x}') + \frac{4a_2+a_3}{2a_2+a_3}\Delta_{\mathbf{x}'}\mathfrak{J}(\mathbf{x}') + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\Delta_{\mathbf{x}'}^2\mathfrak{J}(\mathbf{x}') \right] - \frac{4a_2+a_3}{2a_1(2a_2+a_3)}\mathfrak{J}(\mathbf{x}) \end{cases} \quad (249)$$

Eqs. (244) represent a coupled set of fourth order differential equations. The total number of integration constants is eight. With the substitution (246), it has been possible to decouple the set of equations, but now the differential

order of the single equations is changed. The total differential order is the same, indeed we have an equation of sixth order and another equation of second order, while, previously, we had two equations of fourth order. The number of integration constants is preserved. We can conclude that, with our approach, also the introduction of the new quantities $\tilde{\Phi}$, $\tilde{\Psi}$ does not contradict the paradigm of metric theories of FOG. The price is that now the *r.h.s.* of *tt*-component of the field equation has been modified: there is an additional matter term τ coming from the *ij*-component. In Table III, we show particular cases of (244) for different choices of coupling constants of the theory with the vanishing determinant of transformations (245).

| Case | Choices of constants | Corresponding field equations |
|------|--------------------------------------|--|
| A | $a_2 = 0$ $a_3 = 0$ | $\Delta\Psi = \frac{\mathcal{X}}{2a_1}\rho$ $\Delta\left[\Phi(\mathbf{x}) + \frac{G}{a_1} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{ \mathbf{x}-\mathbf{x}' }\right]\delta_{ij} - \left[\Phi(\mathbf{x}) + \frac{G}{a_1} \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{ \mathbf{x}-\mathbf{x}' }\right]_{,ij} = 0$ |
| B | $a_1 = 0$ $a_3 = 0$ | $\Delta^2(2\Psi - \Phi) = -\frac{\mathcal{X}}{4a_2}\rho$ $\Delta\left[\Delta(2\Psi - \Phi)\right]\delta_{ij} - \left[\Delta(2\Psi - \Phi)\right]_{,ij} = 0$ |
| C | $a_1 = 0$ $a_2 = 0$ | $\Delta^2(\Phi - \Psi) = \frac{\mathcal{X}}{2b_1}\rho$ $\Delta\left[\Delta(\Phi - 3\Psi)\right]\delta_{ij} - \left[\Delta(\Phi - 3\Psi)\right]_{,ij} = 0$ |
| D | $a_3 = -2a_2$ | $2a_2\Delta^2\Psi - a_1\Delta\Psi = -\frac{\mathcal{X}}{2}\rho$ $\nabla^2\left[a_1\Phi(\mathbf{x}) - 2a_2\Delta\Phi(\mathbf{x}) + G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{ \mathbf{x}-\mathbf{x}' }\right]\delta_{ij} - \left[a_1\Phi(\mathbf{x}) - 2a_2\Delta\Phi(\mathbf{x}) + G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{ \mathbf{x}-\mathbf{x}' }\right]_{,ij} = 0$ |
| E | $a_1 = 0$ $a_3 = -4a_2$ | $\Delta^2\Phi = -\frac{\mathcal{X}}{4a_2}\rho$ $\Delta\left[\Delta\Psi\right]\delta_{ij} - \left[\Delta\Psi\right]_{,ij} = 0$ |
| F | $a_1 = 0$ $a_3 = -2a_2$ | $\Delta^2\Psi = -\frac{\mathcal{X}}{4a_2}\rho$ $\Delta\left[\Delta(\Psi - \Phi)\right]\delta_{ij} - \left[\Delta(\Psi - \Phi)\right]_{,ij} = 0$ |
| G | $a_1 = 0$ $a_3 = -\frac{8a_2}{3}$ | $\Delta^2(2\Psi + \Phi) = -\frac{3\mathcal{X}}{4a_2}\rho$ $\Delta\left[\Delta\Phi\right]\delta_{ij} - \left[\Delta\Phi\right]_{,ij} = 0$ |

TABLE III: Explicit form of the field equations for different choices of the coupling constants for which the determinant of transformations (245) vanishes.

A. Green functions for systems with spherical symmetry

As above, we are interested in the solutions of (244) at Newtonian order by using the method of Green functions with spherical symmetry. Let us introduce the radial coordinate $r \doteq |\mathbf{x} - \mathbf{x}'|$; with this choice, the first equation of (248) for $r \neq 0$ becomes

$$2a_3(3a_2 + a_3)\Delta_r^3\Upsilon(r) + (6a_2 + a_3)\Delta_r^2\Upsilon(r) - a_1^2\Delta_r\Upsilon(r) = 0 \quad (250)$$

where $\Delta_r = r^{-2}\partial_r(r^{-2}\partial_r)$ is the radial component of the Laplacian in polar coordinates. The solution of (250) is

$$\Upsilon(r) = K_1 - \frac{1}{r}\left[K_2 + \frac{a_3}{a_1}\left(K_3e^{-\sqrt{-\frac{a_1}{a_3}}r} + K_4e^{\sqrt{-\frac{a_1}{a_3}}r}\right) - \frac{2(3a_2 + a_3)}{a_1}\left(K_5e^{-\sqrt{\frac{a_1}{2(3a_2+a_3)}}r} + K_6e^{\sqrt{\frac{a_1}{2(3a_2+a_3)}}r}\right)\right] \quad (251)$$

where K_i are constants. We note that, if $a_2 = a_3 = 0$, the Green function of the Newtonian Mechanics is recovered. It is the same situation of the Electromagnetism. The integration constants K_i have to be fixed by imposing the boundary conditions at infinity and at the origin. In fact Eq. (251) is a solution of (250) and not of the first equation in (248). A physically consistent solution has to satisfy the condition $\mathcal{T}(\mathbf{x}, \mathbf{x}') \rightarrow 0$, if $|\mathbf{x} - \mathbf{x}'| \rightarrow \infty$, then the constants K_1, K_4, K_6 in Eq.(251) have to vanish. To obtain the conditions on the constants K_2, K_3, K_5 , we consider the Fourier transformation of $\mathcal{T}(\mathbf{x}, \mathbf{x}')$, that is

$$\mathcal{T}(\mathbf{x}, \mathbf{x}') = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \tilde{\mathcal{T}}(\mathbf{k}) e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \quad (252)$$

By putting (252) in the first equation of (248), we obtain

$$\mathcal{T}(\mathbf{x}, \mathbf{x}') = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}}{\frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)}\mathbf{k}^6 - \frac{6a_2+a_3}{2a_2+a_3}\mathbf{k}^4 - \frac{a_1}{2a_2+a_3}\mathbf{k}^2} \quad (253)$$

which, in the case of spherical symmetry, becomes¹¹

$$\mathcal{T}(\mathbf{x}, \mathbf{x}') = - \frac{1}{4\pi^2} \frac{a_1(2a_2+a_3)}{a_3(3a_2+a_3)} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \int_0^\infty \frac{d|\mathbf{k}| \sin|\mathbf{k}||\mathbf{x} - \mathbf{x}'|}{|\mathbf{k}| \left[\mathbf{k}^2 - \frac{a_1}{a_3} \right] \left[\mathbf{k}^2 + \frac{a_1}{2(3a_2+a_3)} \right]} \quad (254)$$

The analytic expression of $\mathcal{T}(\mathbf{x}, \mathbf{x}')$ depends on the nature of the poles of $|\mathbf{k}|$ and on the values of the arbitrary constants a_1, a_2, a_3 . If we define two "masses" m_1 and m_2

$$m_1^2 \doteq -\frac{a_1}{2(3a_2+a_3)}, \quad m_2^2 \doteq \frac{a_1}{a_3} \quad (255)$$

we obtain, in Table IV, the three particular expressions of (254). We note that the definition of mass m_1 is a generalization of (195) (and obviously of (154) and (185)).

| Case | Choices of constants | Green function |
|------|-------------------------------|---|
| A | $a_3 > 0$ $3a_2 + a_3 < 0$ | $\mathcal{T}^A(\mathbf{x}, \mathbf{x}') = \frac{1}{12\pi} \frac{1}{ \mathbf{x} - \mathbf{x}' } \left[\frac{m_1^2 - m_2^2}{m_1^2 m_2^2} + \frac{e^{-m_1 \mathbf{x} - \mathbf{x}' }}{m_1^2} - \frac{e^{-m_2 \mathbf{x} - \mathbf{x}' }}{m_2^2} \right]$ |
| B | $a_3 < 0$ $3a_2 + a_3 > 0$ | $\mathcal{T}^B(\mathbf{x}, \mathbf{x}') = \frac{1}{12\pi} \frac{1}{ \mathbf{x} - \mathbf{x}' } \left[-\frac{m_1^2 - m_2^2}{m_1^2 m_2^2} - \frac{\cos(m_1 \mathbf{x} - \mathbf{x}')}{m_1^2} + \frac{\cos(m_2 \mathbf{x} - \mathbf{x}')}{m_2^2} \right]$ |
| C | $a_3 < 0$ $3a_2 + a_3 < 0$ | $\mathcal{T}^C(\mathbf{x}, \mathbf{x}') = \frac{1}{12\pi} \frac{1}{ \mathbf{x} - \mathbf{x}' } \left[-\frac{m_1^2 + m_2^2}{m_1^2 m_2^2} + \frac{e^{-m_1 \mathbf{x} - \mathbf{x}' }}{m_1^2} + \frac{\cos(m_2 \mathbf{x} - \mathbf{x}')}{m_2^2} \right]$ |

TABLE IV: The complete set of Green functions for (254). It is possible to have a further choice for the scale lengths depending on the other two length scales. If we perform the substitution $m_1^2 \rightleftharpoons -m_2^2$, we obtain a fourth choice. In addition, for a correct Newtonian component, we assumed $a_1 > 0$. In fact when $a_2 = a_3 = 0$ the field Eqs. (243) give the Newtonian limit for $a_1 = 1$.

Besides, we have obtained another Yukawa-like correction to the Newtonian potential related to the squared Ricci tensor correction in the Lagrangian (240). This behavior is strictly linked to the sixth order of (248), which depends

¹¹ we introduced the polar coordinates in the \mathbf{k} -space.

on the coupled form of the system of equations (243). In fact, if we consider the Fourier transform of the potentials Φ and Ψ

$$\Phi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\Phi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \Psi(\mathbf{x}) = \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \hat{\Psi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (256)$$

the solutions are

$$\begin{cases} \Phi(\mathbf{x}) = -\frac{\chi}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{[a_1 + (8a_2 + 3a_3)\mathbf{k}^2] \tilde{\rho}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2(a_1 - a_3\mathbf{k}^2)[a_1 + 2(3a_2 + a_3)\mathbf{k}^2]} \\ \Psi(\mathbf{x}) = -\frac{\chi}{2} \int \frac{d^3\mathbf{k}}{(2\pi)^{3/2}} \frac{[a_1 + (4a_2 + a_3)\mathbf{k}^2] \tilde{\rho}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2(a_1 - a_3\mathbf{k}^2)[a_1 + 2(3a_2 + a_3)\mathbf{k}^2]} \end{cases} \quad (257)$$

where $\tilde{\rho}(\mathbf{k})$ is the Fourier transform of the matter density. We can see that the solutions have the same poles as Eq.(254). Finally, considering the Fourier transform of the point-like source (51), that is $\tilde{\rho}(\mathbf{k}) = \frac{M}{(2\pi)^3}$, the solutions (257) are

$$\begin{cases} \Phi(\mathbf{x}) = -\frac{GM}{a_1} \left(\frac{1}{|\mathbf{x}|} + \frac{1}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{4}{3} \frac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|} \right) \\ \Psi(\mathbf{x}) = -\frac{GM}{a_1} \left(\frac{1}{|\mathbf{x}|} - \frac{1}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{2}{3} \frac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|} \right) \end{cases} \quad (258)$$

where $a_3 \neq 0$ and $3a_2 + a_3 \neq 0$ and $m_1^2, m_2^2 > 0$. The solution for Φ generalizes the third line of (222) in the case of point-like source. In fact, it is interesting to note that, if $a_3 = 0$ ($m_2 \rightarrow \infty$), we have the missing of a scale length (a pole is missed) with only a Yukawa-like term analogously to Electrodynamics. The Green function, in this case, is

$$\tilde{\mathcal{I}}(\mathbf{k})_{a_3=0} = \frac{2a_2}{6a_2\mathbf{k}^4 + a_1\mathbf{k}^2} \quad (259)$$

and Lagrangian (240) becomes $f = a_1 R + a_2 R^2$. Finally the presence of the pole is achieved considering a particular choice of the constants in the theory: $a_3 = -2a_2$. In Table III (Case D), we provide the field equations for this choice and the relative Green function is

$$\tilde{\mathcal{I}}_{(2a_2\nabla^4 - a_1\nabla^2)}(\mathbf{k}) \propto \frac{1}{2a_2\mathbf{k}^4 + a_1\mathbf{k}^2} \quad (260)$$

The spatial behavior of (259) - (260) is the same but the coefficients are different since the theories are different. In conclusion, we need the Green function for the differential operator Δ^2 . The only possible physical choice for the squared Laplacian is

$$\tilde{\mathcal{I}}_{(\Delta^2)}(\mathbf{x} - \mathbf{x}') \propto \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad (261)$$

since the other choice is proportional to $|\mathbf{x} - \mathbf{x}'|$ and cannot to be accepted. Considering the last possibility, we will end up with a force law increasing with distance [17]. In summary, we have shown the general approach to find out solutions of the field equations by using the Green functions. In particular, the vacuum solutions with point-like source have been used to find out directly the potentials, however it remains the important issue to find out solutions for systems with extended matter distributions.

B. Solutions using the Green functions approach

Before investigating the general solution of (243), we want discuss all cases shown in the Table III. Later, we will derive solutions in presence of matter using the Green functions in Table IV.

Specifically, in Table V, we provide solutions in terms of Green functions of the corresponding differential operators coming from the field equations shown in Table III. Case A corresponds to the Newtonian theory and the arbitrary constant a_1 can be absorbed in the definition of matter Lagrangian. The solutions are

$${}_A\Phi(\mathbf{x}) = {}_A\Psi(\mathbf{x}) = -G \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (262)$$

For Case D, instead, we have the field equations of a sort of modified electrodynamic-like representation. The solution can be expressed as follows

$${}_D\Phi(\mathbf{x}) = {}_D\Psi(\mathbf{x}) = -G \int d^3\mathbf{x}' \left[\frac{1 - e^{-\sqrt{\frac{a_1}{2a_2}}|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} \right] \rho(\mathbf{x}') \quad (263)$$

The solutions make sense only if $a_1/a_2 > 0$, then we can introduce a new scale-length. A particular expression of (263), for a fixed matter density $\rho(\mathbf{x})$, will be found in a more general context in the next section. Nevertheless these two cases are the only ones which exhibit the Newtonian limit (obviously the first one!), while for the remaining cases there are serious problems with divergences and incompatibilities. In fact, Case B presents an incompatibility between the solution obtained from the tt -component and the one from the ij -component. The incompatibility can be removed if we consider, as the Green function for the differential operator ∇^4 , the trivial solution: $\mathcal{G}_{(\Delta^2)}|_B = \text{const}$. With this choice, the arbitrary integration constant Φ_0 can be interpreted as $-GM$. However another problem remains: namely the divergence at the origin. The interpretation of the constant Φ_0 as a total mass and not as a generic integral $\int d^3\mathbf{x}' \rho(\mathbf{x}')$ does not avoid the singularity. Finally, the solution

$${}_B\Psi(\mathbf{x}) - {}_B\Phi(\mathbf{x}) = -\frac{GM}{|\mathbf{x} - \mathbf{x}'|} \quad (264)$$

holds only in vacuum. Before continuing the analysis of the various cases, the term $\int d^3\mathbf{x}' \mathcal{T}_{(\Delta^2)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ has to be discussed for the choice (261). A generic field equation with Δ^2 (from Table III) is

$$\Delta_{\mathbf{x}}^2 \Phi(\mathbf{x}) \propto \Delta_{\mathbf{x}}^2 \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \propto \Delta_{\mathbf{x}} \rho(\mathbf{x}) \neq 4\pi \rho(\mathbf{x}) \quad (265)$$

from which the only consistent possibility is to set $\rho(\mathbf{x}) = 0$. In the remaining cases, we can only consider vacuum solutions.

By considering solutions (249) with the Green function $\mathcal{T}^A(\mathbf{x}, \mathbf{x}')$ from Table IV and by assuming $\mathfrak{J}(\mathbf{x}) = 0^{12}$, we have

$$\begin{cases} {}_A\Phi(\mathbf{x}) = \mathcal{X} \frac{(8a_2+3a_3)\Delta_{\mathbf{x}}-a_1}{2a_1(2a_2+a_3)} \int d^3\mathbf{x}' \mathcal{T}^A(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') \\ {}_A\Psi(\mathbf{x}) = \mathcal{X} \frac{(4a_2+a_3)\Delta_{\mathbf{x}}-a_1}{2a_1(2a_2+a_3)} \int d^3\mathbf{x}' \mathcal{T}^A(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}') \end{cases} \quad (266)$$

After some algebraic calculations we get, for a spherically symmetric matter source (203), the internal and external solutions for Φ

$$\begin{cases} {}_A\Phi_{in}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{8m_1^2+m_2^2(2+3m_1^2\xi^2)}{2m_1^2m_2^2\xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} - \frac{e^{-m_1\xi}(1+m_1\xi)}{m_1^2\xi^3} \frac{\sinh m_1|\mathbf{x}|}{m_1|\mathbf{x}|} + 4 \frac{e^{-m_2\xi}(1+m_2\xi)}{m_2^2\xi^3} \frac{\sinh m_2|\mathbf{x}|}{m_2|\mathbf{x}|} \right] \\ {}_A\Phi_{out}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{1}{|\mathbf{x}|} + \frac{F(m_1\xi)}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{4F(m_2\xi)}{3} \frac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|} \right] \end{cases} \quad (267)$$

¹² We have to note that our working hypothesis, $\mathfrak{J}(\mathbf{x}) = 0$, is not particular, since when we considered the Hilbert-Einstein Lagrangian to give the Newtonian solution, we imposed an analogous condition.

| Case | Solutions | Newtonian behavior |
|------|--|--------------------|
| A | $\Phi(\mathbf{x}) = \Psi(\mathbf{x}) = -\frac{G}{a_1} \int d^3 \mathbf{x}' \frac{\rho(\mathbf{x}')}{ \mathbf{x} - \mathbf{x}' }$ | yes |
| B | $2\Psi(\mathbf{x}) - \Phi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} }$ $2\Psi(\mathbf{x}) - \Phi(\mathbf{x}) = -\frac{2\pi G}{a_2} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ | no |
| C | $\Phi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} } + \frac{6\pi G}{a_3} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ $\Psi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} } + \frac{2\pi G}{a_3} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ | no |
| D | $\Phi(\mathbf{x}) = -4\pi G \int d^3 \mathbf{x}' \mathcal{T}_{(2a_2 \nabla^4 - a_1 \nabla^2)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ $\Psi(\mathbf{x}) = -4\pi G \int d^3 \mathbf{x}' \mathcal{T}_{(2a_2 \nabla^4 - a_1 \nabla^2)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ | yes |
| E | $\Phi(\mathbf{x}) = -\frac{2\pi G}{a_2} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ $\Psi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} }$ | no |
| F | $\Phi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} } - \frac{2\pi G}{a_2} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ $\Psi(\mathbf{x}) = -\frac{2\pi G}{a_2} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ | no |
| G | $\Phi(\mathbf{x}) = \frac{\Phi_0}{ \mathbf{x} }$ $\Psi(\mathbf{x}) = -\frac{1}{2} \frac{\Phi_0}{ \mathbf{x} } - \frac{3\pi G}{a_2} \int d^3 \mathbf{x}' \mathcal{T}_{(\nabla^4)}(\mathbf{x}, \mathbf{x}') \rho(\mathbf{x}')$ | no |

TABLE V: Solutions of the field equations in Table III. Solutions are found by setting $\mathcal{J} = 0$ in the ij -component of the field Eqs. (248). Solutions are displayed in terms of the Green functions. Φ_0 is a generic integration constant.

representing a generalization of metric potential (214). It is easy to check that when $a_3 \rightarrow 0$ (*i.e.* $m_2 \rightarrow \infty$), we get exactly Eq. (214). If we consider the limit of point-like source, we get the first line of (258).

For the sake of completeness, let us derive solutions for the other Green functions. Starting from Case B in Table IV, yet we have the internal and external potentials

$$\left\{ \begin{array}{l} {}_B\Phi_{in}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{m_2^2(2+3m_1^2\xi^2)-8m_1^2}{2m_1^2m_2^2\xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} - \frac{\cos m_1\xi + m_1\xi \sin m_1\xi}{m_1^2\xi^3} \frac{\sin m_1|\mathbf{x}|}{m_1|\mathbf{x}|} + 4 \frac{\cos m_2\xi + m_2\xi \sin m_2\xi}{m_2^2\xi^3} \frac{\sin m_2|\mathbf{x}|}{m_2|\mathbf{x}|} \right] \\ {}_B\Phi_{out}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{1}{|\mathbf{x}|} + \frac{G(m_1\xi) \cos m_1|\mathbf{x}|}{3|\mathbf{x}|} - \frac{4G(m_2\xi) \cos m_2|\mathbf{x}|}{3|\mathbf{x}|} \right] \end{array} \right. \quad (268)$$

and its limit of point-like source is

$$\lim_{\xi \rightarrow 0} {}_B\Phi_{out}(|\mathbf{x}|) = -\frac{GM}{a_1} \left[\frac{1}{|\mathbf{x}|} - \frac{1}{6} \frac{\cos m_1|\mathbf{x}|}{|\mathbf{x}|} + \frac{2}{3} \frac{\cos m_2|\mathbf{x}|}{|\mathbf{x}|} \right] \quad (269)$$

Finally for Case C in Table IV, we have

$$\begin{cases} {}_C\Phi_{in}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{m_2^2(2+3m_1^2\xi^2)+8m_1^2}{2m_1^2m_2^2\xi^3} - \frac{|\mathbf{x}|^2}{2\xi^3} - \frac{e^{-m_1\xi}(1+m_1\xi)}{m_1^2\xi^3} \frac{\sinh m_1|\mathbf{x}|}{m_1|\mathbf{x}|} - 4 \frac{\cos m_2\xi+m_2\xi \sin m_2\xi}{m_2^2\xi^3} \frac{\sin m_2|\mathbf{x}|}{m_2|\mathbf{x}|} \right] \\ {}_C\Phi_{out}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{1}{|\mathbf{x}|} + \frac{F(m_1\xi)}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} + \frac{4G(m_2\xi)}{3} \frac{\cos m_2|\mathbf{x}|}{|\mathbf{x}|} \right] \end{cases} \quad (270)$$

which in the point-like limit becomes

$$\lim_{\xi \rightarrow 0} {}_C\Phi_{out}(\mathbf{x}) = -\frac{GM}{a_1} \left[\frac{1}{|\mathbf{x}|} + \frac{1}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{4 \cos m_2|\mathbf{x}|}{3 |\mathbf{x}|} \right] \quad (271)$$

Similar relations are found for Ψ .

These results mean that, for suitable distance scales, the validity of Gauss theorem is restored and the theory agrees with the standard Newtonian limit of GR.

IX. THE NEWTONIAN LIMIT OF $f(X, Y, Z)$ -GRAVITY

We conclude this review providing the Newtonian limit of a generic $f(X, Y, Z)$ -gravity [59]. By considering the paradigm of Newtonian limit (98), the curvature invariants X, Y, Z become

$$\begin{cases} X \sim X^{(2)} + X^{(4)} + \dots \\ Y \sim Y^{(4)} + Y^{(6)} + \dots \\ Z \sim Z^{(4)} + Z^{(6)} + \dots \end{cases} \quad (272)$$

and the function f can be developed as

$$f(X, Y, Z) \sim f(0) + f_X(0)X^{(2)} + \frac{1}{2}f_{XX}(0)X^{(2)^2} + f_X(0)X^{(4)} + f_Y(0)Y^{(4)} + f_Z(0)Z^{(4)} + \dots \quad (273)$$

and analogous relations for partial derivatives of f are obtained. From the lowest order of field Eqs. (21), we have

$$f(0) = 0 \quad (274)$$

this means that not only in $f(R)$ -gravity [36, 45] but also in $f(X, Y, Z)$ -gravity, a missing cosmological component in the action (18) implies that the space-time is asymptotically Minkowskian. Eqs. (21) and (22) at $\mathcal{O}(2)$ -order become¹³

$$\begin{cases} H_{tt}^{(2)} = f_X(0)R_{tt}^{(2)} - [f_Y(0) + 4f_Z(0)]\Delta R_{tt}^{(2)} - \frac{f_X(0)}{2}X^{(2)} - [f_{XX}(0) + \frac{f_Y(0)}{2}]\Delta X^{(2)} = \mathcal{X}T_{tt}^{(0)} \\ H_{ij}^{(2)} = f_X(0)R_{ij}^{(2)} - [f_Y(0) + 4f_Z(0)]\Delta R_{ij}^{(2)} + \frac{f_X(0)}{2}X^{(2)}\delta_{ij} + [f_{XX}(0) + \frac{f_Y(0)}{2}]\Delta X^{(2)}\delta_{ij} - f_{XX}(0)X^{(2)}_{,ij} + \\ \quad + [f_Y(0) + 4f_Z(0)]R_{mi,jm}^{(2)} + f_Y(0)R_{mj,im}^{(2)} = 0 \\ H^{(2)} = -f_X(0)X^{(2)} - [3f_{XX}(0) + 2f_Y(0) + 2f_Z(0)]\Delta X^{(2)} = \mathcal{X}T^{(0)} \end{cases} \quad (275)$$

By introducing the quantities

¹³ We used the properties $R_{\alpha\beta}{}^{;\alpha\beta} = \frac{1}{2}\square R$ and $R_{\mu}{}^{\alpha\beta}{}_{;\nu;\alpha\beta} = R_{\mu}{}^{\alpha}{}_{;\nu\alpha} - \square R_{\mu\nu}$.

$$\begin{cases} m_1^2 \doteq -\frac{f_X(0)}{3f_{XX}(0)+2f_Y(0)+2f_Z(0)} \\ m_2^2 \doteq \frac{f_X(0)}{f_Y(0)+4f_Z(0)} \end{cases} \quad (276)$$

we get three differential equations for curvature invariant $X^{(2)}$, tt - and ij -component of Ricci tensor $R_{\mu\nu}^{(2)}$

$$\begin{cases} (\Delta - m_2^2)R_{tt}^{(2)} + \left[\frac{m_2^2}{2} - \frac{m_1^2+2m_2^2}{6m_1^2}\Delta \right] X^{(2)} = -\frac{m_2^2\mathcal{X}}{f_X(0)} T_{tt}^{(0)} \\ (\Delta - m_2^2)R_{ij}^{(2)} + \left[\frac{m_1^2-m_2^2}{3m_1^2}\partial_{ij}^2 - \left(\frac{m_2^2}{2} - \frac{m_1^2+2m_2^2}{6m_1^2}\Delta \right) \delta_{ij} \right] X^{(2)} = 0 \\ (\Delta - m_1^2)X^{(2)} = \frac{m_1^2\mathcal{X}}{f_X(0)} T^{(0)} \end{cases} \quad (277)$$

We note that the definitions (276) are a generalization of (255). While the interpretation of m_1 is clear from the trace field equation of $f(R)$ -gravity, now also m_2 is clear. In fact, from the two first lines of (277) the Ricci tensor has a characteristic length (m_2^{-1}) .

The solution for curvature invariant $X^{(2)} = R^{(2)}$ at third line of (277) is

$$X^{(2)}(t, \mathbf{x}) = \frac{m_1^2\mathcal{X}}{f_X(0)} \int d^3\mathbf{x}' \mathcal{G}_1(\mathbf{x}, \mathbf{x}') T^{(0)}(t, \mathbf{x}') \quad (278)$$

where $\mathcal{G}_1(\mathbf{x}, \mathbf{x}')$ is the Green function of the field operator $\Delta - m_1^2$ (this solution is formally equal to (194)). The solution for $g_{tt}^{(2)}$, by (103), is the following

$$g_{tt}^{(2)}(t, \mathbf{x}) = \frac{1}{2\pi} \int d^3\mathbf{x}' d^3\mathbf{x}'' \frac{\mathcal{G}_2(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{m_2^2\mathcal{X}}{f_X(0)} T_{tt}^{(0)}(t, \mathbf{x}'') - \frac{(m_1^2 + 2m_2^2)\mathcal{X}}{6f_X(0)} T^{(0)}(t, \mathbf{x}'') + \frac{m_2^2 - m_1^2}{6} X^{(2)}(t, \mathbf{x}'') \right] \quad (279)$$

where $\mathcal{G}_2(\mathbf{x}, \mathbf{x}')$ is the Green function of field operator $\Delta - m_2^2$. The general solution for $g_{ij}^{(2)}$ from Eqs. (277), if we consider the expression (116) (gauge harmonic condition), is

$$g_{ij}^{(2)}|_{HG} = \frac{1}{2\pi} \int d^3\mathbf{x}' d^3\mathbf{x}'' \frac{\mathcal{G}_2(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}'|} \left[\frac{m_1^2 - m_2^2}{3m_1^2} \partial_{i''j''}^2 - \left(\frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \Delta_{\mathbf{x}''} \right) \delta_{ij} \right] X^{(2)}(\mathbf{x}'') \quad (280)$$

If we choose the metric (242), from (103) we have $R_{ij}^{(2)} = \Delta\Psi\delta_{ij} + (\Psi - \Phi)_{,ij}$ and the second field equation of (277) becomes

$$\begin{cases} \Delta\Psi = \int d^3\mathbf{x}' \mathcal{G}_2(\mathbf{x}, \mathbf{x}') \left(\frac{m_2^2}{2} - \frac{m_1^2+2m_2^2}{6m_1^2} \Delta_{\mathbf{x}'} \right) X^{(2)}(\mathbf{x}') \\ (\Phi - \Psi)_{,ij} = \frac{m_1^2 - m_2^2}{3m_1^2} \int d^3\mathbf{x}' \mathcal{G}_2(\mathbf{x}, \mathbf{x}') X_{,i'j'}^{(2)}(\mathbf{x}') \end{cases} \quad (281)$$

Then the general solution for $g_{ij}^{(2)}$ from (277), *without gauge conditions* and by using the first line of (281), is

$$g_{ij}^{(2)} = 2\Psi\delta_{ij} = -\frac{\delta_{ij}}{2\pi} \int d^3\mathbf{x}' d^3\mathbf{x}'' \frac{\mathcal{G}_2(\mathbf{x}', \mathbf{x}'')}{|\mathbf{x} - \mathbf{x}'|} \left(\frac{m_2^2}{2} - \frac{m_1^2 + 2m_2^2}{6m_1^2} \Delta_{\mathbf{x}''} \right) X^{(2)}(\mathbf{x}'') \quad (282)$$

and the second line of (281) is only a constraint condition for metric potentials. In fact from its trace, we have

$$\Delta(\Phi - \Psi) = \frac{m_1^2 - m_2^2}{3m_1^2} \int d^3\mathbf{x}' \mathcal{G}_2(\mathbf{x}, \mathbf{x}') \Delta_{\mathbf{x}'} X^{(2)}(\mathbf{x}') \quad (283)$$

and we can affirm that only in GR the metric potentials Φ and Ψ are equals.

Let us consider the point-like source (51). If we choose $m_1^2 > 0$ and $m_2^2 > 0$, the curvature invariant $X^{(2)}$ (278) and the metric potentials Φ (279) and Ψ (282) are

$$\begin{cases} X^{(2)} = -\frac{r_g m_1^2}{f_X(0)} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} \\ \Phi = -\frac{GM}{f_X(0)} \left[\frac{1}{|\mathbf{x}|} + \frac{1}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{4}{3} \frac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|} \right] \\ \Psi = -\frac{GM}{f_X(0)} \left[\frac{1}{|\mathbf{x}|} - \frac{1}{3} \frac{e^{-m_1|\mathbf{x}|}}{|\mathbf{x}|} - \frac{2}{3} \frac{e^{-m_2|\mathbf{x}|}}{|\mathbf{x}|} \right] \end{cases} \quad (284)$$

The modified gravitational potential of $f(R)$ -gravity is further modified by the presence of functions of $R_{\alpha\beta}R^{\alpha\beta}$ (as in the *Quadratic Lagrangian*-theory) and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$. The curvature invariant $X^{(2)}$ presents a massive propagation and when $f(X, Y, Z) \rightarrow f(R)$ we find the mass definition $m^2 = -f'(R=0)/3f''(R=0)$ given in (195). In this case, the propagation mode with m_2 disappears. Obviously the two expressions for gravitational potentials in (284) satisfy the constraint condition (283). In FIGs. 11 and 12 we sketch the spatial behavior of metric potentials for some values of parameters m_1 and m_2 .

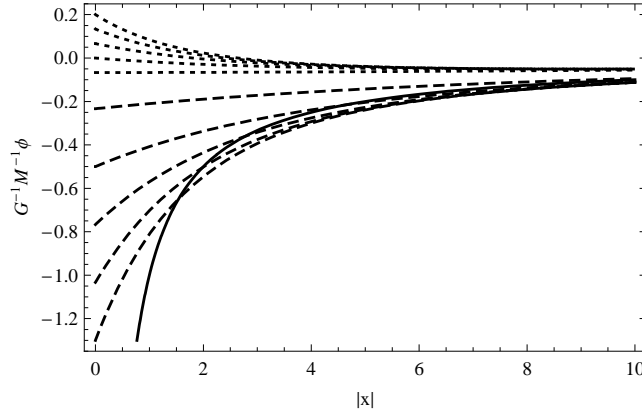


FIG. 11: Plot of metric potential Φ in Eqs. (284). $m_2 = \zeta m_1$ and $m_1 = 0.1$ (dotted line), $m_1 = \zeta m_2$ and $m_2 = 0.1$ (dashed line). The behavior of GR is shown by the solid line. The dimensionless quantity ζ runs between $0 \div 10$. We set $f_X(0) = 1$.

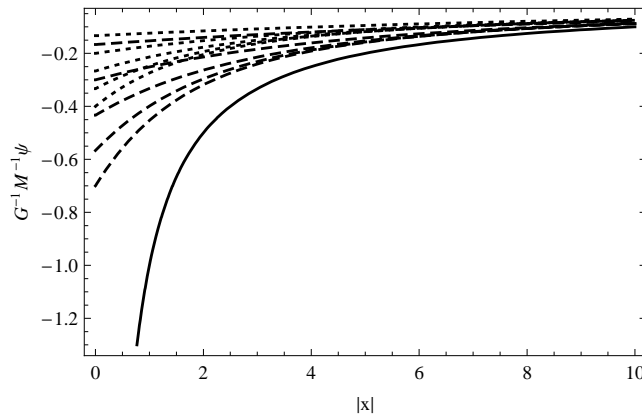


FIG. 12: Plot of metric potential Ψ in Eqs. (284). $m_2 = \zeta m_1$ and $m_1 = 0.1$ (dotted line), $m_1 = \zeta m_2$ and $m_2 = 0.1$ (dashed line). The behavior of GR is shown by the solid line. The dimensionless quantity ζ runs between $0 \div 10$. We set $f_X(0) = 1$.

The same outcome has been found for *Quadratic Lagrangian* if we identify $a_1 = f_X(0)$. We can affirm, then, the Newtonian limit of any $f(X, Y, Z)$ -Gravity can be reinterpreted by introducing the *Quadratic Lagrangian* and the coefficients have to satisfy the following relations

$$a_1 = f_X(0), \quad a_2 = \frac{1}{2}f_{XX}(0) - f_Z(0), \quad a_3 = f_Y(0) + 4f_Z(0) \quad (285)$$

if we want the same definition of parameters in (255) and (276).

A first considerations about (285) concerns the characteristic lengths induced by $f(X, Y, Z)$ -theory. The second length m_2^{-1} is originated by the presence, in the Lagrangian, of squared Ricci and Riemann tensors, but also a theory containing only squared Ricci tensor shows the same outcome. Obviously the same is valid also with the only squared Riemann tensor. Such terms give rise to massive gravitational modes and then to the possibility of massive gravitational waves (see [61] and references therein).

A second consideration concerns the Gauss - Bonnet invariant defined by the relation $G_{GB} = X^2 - 4Y + Z$ [60]. In fact the induced field equations satisfy, in four dimensions, the condition

$$H_{\mu\nu}^{GB} = H_{\mu\nu}^{X^2} - 4H_{\mu\nu}^Y + H_{\mu\nu}^Z = 0 \quad (286)$$

and, by substituting them at Newtonian level ($H_{tt}^Z \sim -4\Delta R_{tt}^{(2)}$) in Eqs. (21), we find the field equations (ever at Newtonian Level) of *Quadratic Lagrangian*.

A third and last consideration is about the solutions (284). When we perform the limit in the origin $|\mathbf{x}| = 0$ we have no divergence. In fact we find

$$\lim_{|\mathbf{x}| \rightarrow 0} \Phi \propto \frac{m_1 - 4m_2}{3}, \quad \lim_{|\mathbf{x}| \rightarrow 0} \Psi \propto -\frac{m_1 + 2m_2}{3} \quad (287)$$

and only if we remove, in the action (18), the dependence on the squared Ricci or squared Riemann tensors, we get the divergence of GR. For a physical interpretation of solution (284), we must impose the condition $m_1 - 4m_2 < 0$ to have a potential well with a negative minimum in $|\mathbf{x}| = 0$ and $m_1 < m_2$ to have a negative profile of potential (see FIG. 11). Then, if we suppose $f_X(0) > 0$, we get a constraint on the derivatives of f with respect to curvature invariants, that is

$$f_{XX}(0) + f_Y(0) + 2f_Z(0) < 0 \quad (288)$$

In the case of $f(R)$ -gravity ($f_Y(0) = f_Z(0) = 0$) we reobtain the same condition between the first and second derivatives of $f(R)$.

X. CONCLUSIONS

The weak field limit is a crucial issue that has to be addressed in any relativistic theory of gravity. It is also the test bed of such theories in order to compare them with the well-founded experimental results of GR, at least at Solar system level.

In this review paper, we have considered the problem of weak field limit of Fourth Order Gravity, that is of gravitational theories where curvature invariants, a part the standard Ricci scalar, linear in the Hilbert-Einstein action, are taken into account. In particular, we have analyzed the Newtonian and the post-Newtonian limits of theories involving non-linear combinations of Ricci scalar, Ricci tensor and Riemann tensor. The calculations have been essentially developed in the so-called Jordan frame but we have also considered the conformal transformations and the possible shortcomings emerging in carrying the weak field limit results in the Einstein frame without an appropriate interpretation of post-Newtonian parameters.

The general feature that emerges from the weak field limit is that corrections to the Newtonian potential naturally come out. These corrections are Yukawa-like terms bringing characteristic masses and lengths. Conversely, the standard Newtonian potential is just a feature emerging in the particular case $f(R) = R$. These characteristic masses (and lengths) come out as combinations of the parameters of the theory and fix the scales where corrections become relevant.

These results open new intriguing possibilities since accurate measurements of PPN parameters could confirm or rule out these theories in view of the forthcoming space experiments as GAIA and GAME (see [62–64] for a detailed review of the status of art).

On the other hand, it is well-known that the new features related to extended theories of gravity could have interesting applications in other fields of astrophysics as galactic dynamics [65], large scale structure [66] and cosmology [67] in order to address dark matter and dark energy issues. The fact that such "dark" structures have not been definitely discovered at fundamental quantum scales but operate at large astrophysical (infra-red scales) could be due to these corrections to the Newtonian potential which can be hardly detected at laboratory or Solar System scales. Finally, the presence of unavoidable light massive modes could open new opportunities also for the gravitational waves detection of experiments like VIRGO, LIGO and the forthcoming LISA [61].

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